The result found on the page 1407 - the "major result of this paper" - has the following form for the "pressure part" for the mixture pressure

$$p_m \cong \beta_C(\mathbf{x})\langle p\rangle + \left(1+\frac{a^2}{10}\nabla^2\right)(n\nu\overline{p}^e) + \\ +\frac{a^2}{5}\nabla\cdot\left(n\overline{\int\limits_{|\mathbf{r}|=a} dS_r\,(-p_C)\,\mathbf{n}}\right) + \\ +\frac{a^2}{14}\nabla\nabla\cdot\left(n\overline{\int\limits_{|\mathbf{r}|=a} dS_r\left(\mathbf{nn}-\frac{1}{3}\mathbf{I}\right)p_C}\right)+\ldots \tag{16}$$

where $\overline{p}^e$ is the surface-average of the continuous-phase pressure over the particle surface

$$\overline{p}^e = \frac{1}{4\pi a^2}\int\limits_{|\mathbf{r}|=a} dS_r\,(p_C)\,. \tag{17}$$

The methodology and mathematical expressions are based on a perturbation expansion of the stress tensor and the sign $\cong$ should be used instead of $=$. It is not clear that there will ever be a way to evaluate the level of approximation.

It will be extremely difficult for the authors to properly treat terms like $\nabla\cdot(\beta_C\langle\boldsymbol{\sigma}_C\rangle)$ and $\nabla\cdot\left(\overleftrightarrow{\boldsymbol{\Sigma}}_a\right)$, the averaged stresses, in their averaged equations. These terms are the most complicated part of the right hand side of the momentum equation. They present this most complicated part in the averaged right hand side of the momentum equation by

$$\nabla\cdot(\beta_C\langle\boldsymbol{\sigma}_C\rangle + \beta_D L[\boldsymbol{\sigma}_C]) - nA[\boldsymbol{\sigma}_C] = \nabla\cdot\left[-(p_m+q_m)\,\mathbf{I}+\overleftrightarrow{\mathbf{S}}+\overleftrightarrow{\mathbf{A}}_P\right] -$$

$$-\frac{\beta_D}{\nu}A + \frac{a^2}{10}\left[(\nabla n)\times(\nabla\times A)+n\nabla(\nabla\cdot A)\right]+\ldots, \quad (\mathbf{80}),$$

where the traceless symmetric component $\overleftrightarrow{\mathbf{S}}$ is

$$\overleftrightarrow{\mathbf{S}} = \beta_{\mathsf{C}}\left(\langle\boldsymbol{\sigma}_{\mathsf{C}}\rangle + \langle p_{\mathsf{C}}\rangle\mathbf{I}\right) + \left(1+\frac{a^2}{14}\nabla^2\right)(n\mathbf{t}^{\mathsf{s}}) + \nabla\cdot(n\mathbf{s}^{\mathsf{s}}) + \nabla\nabla\colon (n\mathbf{r}^{\mathsf{s}}) - $$

$$-\frac{a^2}{10}n\left[\nabla A + (\nabla A)^{\mathsf{T}} - \frac{2}{3}\mathbf{I}(\nabla\cdot A)\right] + ..., \quad (\mathbf{76}),$$

and antisymmetric component $\overleftrightarrow{\mathbf{A}}_{\mathsf{P}}$ is

$$\overleftrightarrow{\mathbf{A}}_{\mathsf{P\,j\,i}} = \epsilon_{\mathsf{ijk}}\frac{1}{2}\left[\overline{n\int_{|\mathbf{r}|=\mathsf{a}} dS_{\mathsf{r}}\,(\boldsymbol{\sigma}_{\mathsf{C}}\cdot\mathbf{n})\times\mathbf{r}} - \frac{1}{2}\nabla\cdot\left(\overline{n\int_{|\mathbf{r}|=\mathsf{a}} dS_{\mathsf{r}}\mathbf{r}\,((\boldsymbol{\sigma}_{\mathsf{C}}\cdot\mathbf{n})\times\mathbf{r})}\right) + \right.$$

$$\left. + \frac{1}{2}\nabla\nabla\colon\left(\overline{n\int_{|\mathbf{r}|=\mathsf{a}} dS_{\mathsf{r}}\mathbf{r}\mathbf{r}\,((\boldsymbol{\sigma}_{\mathsf{C}}\cdot\mathbf{n})\times\mathbf{r})}\right)\right] + ..., \quad (\mathbf{78}),$$

and "the isotropic part of the viscous stress is"

$$q_{\mathsf{m}} = \frac{a^2}{5}\partial_{\mathsf{k}}\left(nA^*_{\mathsf{k}}\right) - \frac{a^2}{14}\partial_{\mathsf{k}}\partial_{\mathsf{l}}\left(nt^{\mathsf{s}}_{\mathsf{kl}}\right) + \frac{a^2}{15}n\nabla\cdot A - \partial_{\mathsf{k}}\left(ns^{\mathsf{i}}_{\mathsf{kmm}}\right) - \partial_{\mathsf{l}}\partial_{\mathsf{k}}\left(ns^{\mathsf{i}}_{\mathsf{kmm}}\right), \quad (79).$$

On page 1417 they give the final momentum equation for the continuous phase,

$$\rho_{\mathsf{C}}\beta_{\mathsf{C}}\frac{\partial}{\partial t}\left(\langle\mathbf{u}_{\mathsf{C}}\rangle\right) + \rho_{\mathsf{C}}\beta_{\mathsf{C}}\langle\mathbf{u}_{\mathsf{C}}\rangle\cdot\nabla\langle\mathbf{u}_{\mathsf{C}}\rangle = -\beta_{\mathsf{C}}\nabla\cdot\left(-p_{\mathsf{m}}\mathbf{I} + \boldsymbol{\Sigma}_{\mathsf{C}}\right)\beta_{\mathsf{D}}\mathbf{f} +$$

$$+\rho_{\mathsf{D}}\nabla\cdot\left(\beta_{\mathsf{C}}\mathbf{M}_{\mathsf{C}}\right) - \beta_{\mathsf{C}}\nabla\psi_{\mathsf{C}} + \frac{a^2}{10}\left[(\nabla n)\times(\nabla\times A) + n\nabla(\nabla\cdot A)\right] + ..., (126), \qquad (18)$$

where the kinematic fluctuations induced stress tensor $\mathbf{M}_{\mathsf{C}}$ is given by

$$\mathbf{M}_{\mathsf{C}} = \langle\mathbf{u}_{\mathsf{C}}\rangle\langle\mathbf{u}_{\mathsf{C}}\rangle - \langle\mathbf{u}_{\mathsf{C}}\mathbf{u}_{\mathsf{C}}\rangle, \quad (125),$$

and where the continuous phase viscous contribution to the mixture stress $\boldsymbol{\Sigma}_{\mathsf{C}}$ is

$$\boldsymbol{\Sigma}_{\mathsf{C}} = -q_{\mathsf{m}}\mathbf{I} + \overleftrightarrow{\mathbf{S}} + \overleftrightarrow{\mathbf{A}}_{\mathsf{P}}$$

where q_{m} is the isotropic part of the viscous stress (which in turn is the another incredibly complicated expression (see 79)), $\overleftrightarrow{\mathbf{S}}$ – (*equ* 78) and $\overleftrightarrow{\mathbf{A}}_{\mathsf{P}}$ – (*equ* 77) are even more complicated expressions for the symmetric and antisymmetric components of the stress tensor; and $\mathbf{f}$ is

$$\mathbf{f} = \frac{1}{\boldsymbol{\nu}} A - \boldsymbol{\nabla} \cdot \left(-p_{\mathsf{m}} \mathbf{I} + \boldsymbol{\Sigma}_{\mathsf{C}} \right), \quad (123).$$

If one substitutes these and other expressions into the final momentum equation, the following results:

$$\rho_{\mathsf{C}} \beta_{\mathsf{C}} \frac{\partial}{\partial t} \left(\langle \mathbf{u}_{\mathsf{C}} \rangle \right) + \rho_{\mathsf{C}} \beta_{\mathsf{C}} \langle \mathbf{u}_{\mathsf{C}} \rangle \cdot \nabla \langle \mathbf{u}_{\mathsf{C}} \rangle = -\beta_{\mathsf{C}} \nabla \cdot \left[- \left(p_{\mathsf{m}} + q_{\mathsf{m}} \right) \mathbf{I} + \right.$$

$$+ \beta_{\mathsf{C}} \left(\langle \boldsymbol{\sigma}_{\mathsf{C}} \rangle + \langle p_{\mathsf{C}} \rangle \mathbf{I} \right) + \left(1 + \frac{a^2}{14} \nabla^2 \right) \left(n \mathbf{t}^{\mathsf{S}} \right) + \nabla \cdot \left(n \mathbf{s}^{\mathsf{S}} \right) + \nabla \nabla : \left(n \mathbf{r}^{\mathsf{S}} \right) -$$

$$- \frac{a^2}{10} n \left[\nabla \overline{\int_{|\mathbf{r}|=\mathsf{a}} dS_{\mathsf{r}} \boldsymbol{\sigma}_{\mathsf{C}} \left(\mathbf{x} + \mathbf{r} | \, \mathbf{x}, N-1 \right) \cdot \mathbf{n}} + \right.$$

$$+ \left(\nabla \overline{\int_{|\mathbf{r}|=\mathsf{a}} dS_{\mathsf{r}} \boldsymbol{\sigma}_{\mathsf{C}} \left(\mathbf{x} + \mathbf{r} | \, \mathbf{x}, N-1 \right) \cdot \mathbf{n}} \right)^{\mathsf{T}} -$$

$$\left. - \frac{2}{3} \mathbf{I} \left(\nabla \cdot \overline{\int_{|\mathbf{r}|=\mathsf{a}} dS_{\mathsf{r}} \boldsymbol{\sigma}_{\mathsf{C}} \left(\mathbf{x} + \mathbf{r} | \, \mathbf{x}, N-1 \right) \cdot \mathbf{n}} \right) \right] +$$

$$+ \epsilon_{\mathsf{ijk}} \frac{1}{2} \left[\overline{n \int_{|\mathbf{r}|=\mathsf{a}} dS_{\mathsf{r}} \left(\boldsymbol{\sigma}_{\mathsf{C}} \cdot \mathbf{n} \right) \times \mathbf{r}} - \frac{1}{2} \nabla \cdot \left(\overline{n \int_{|\mathbf{r}|=\mathsf{a}} dS_{\mathsf{r}} \mathbf{r} \left(\left(\boldsymbol{\sigma}_{\mathsf{C}} \cdot \mathbf{n} \right) \times \mathbf{r} \right)} \right) + \right.$$

$$\left. \left. + \frac{1}{2} \nabla \nabla : \left(\overline{n \int_{|\mathbf{r}|=\mathsf{a}} dS_{\mathsf{r}} \mathbf{r} \mathbf{r} \left(\left(\boldsymbol{\sigma}_{\mathsf{C}} \cdot \mathbf{n} \right) \times \mathbf{r} \right)} \right) \right] \right] *$$

$$* \left[\beta_{\mathsf{D}} \frac{1}{\boldsymbol{\nu}} \overline{\int_{|\mathbf{r}|=\mathsf{a}} dS_{\mathsf{r}} \boldsymbol{\sigma}_{\mathsf{C}} \left(\mathbf{x} + \mathbf{r} | \, \mathbf{x}, N-1 \right) \cdot \mathbf{n}} - \beta_{\mathsf{D}} \left[\boldsymbol{\nabla} \cdot \left(- \left(p_{\mathsf{m}} + q_{\mathsf{m}} \right) \mathbf{I} + \right. \right. \right. +$$

$$+\beta_{\mathsf{C}}\left(\left\langle \boldsymbol{\sigma}_{\mathsf{C}}\right\rangle + \left\langle p_{\mathsf{C}}\right\rangle \mathbf{I}\right) + \left(1+\frac{a^2}{14}\nabla^2\right)\left(n\mathbf{t}^{\mathsf{s}}\right) + \nabla\cdot\left(n\mathbf{s}^{\mathsf{s}}\right) + \nabla\nabla\colon \left(n\mathbf{r}^{\mathsf{s}}\right) -$$

$$-\frac{a^2}{10}n\left[\nabla\overline{\int\limits_{|\mathbf{r}|=\mathsf{a}} dS_{\mathsf{r}}\boldsymbol{\sigma}_{\mathsf{C}}\left(\mathbf{x}+\mathbf{r}|\,\mathbf{x},N-1\right)\cdot\mathbf{n}} + \right.$$

$$+\left(\nabla\overline{\int\limits_{|\mathbf{r}|=\mathsf{a}} dS_{\mathsf{r}}\boldsymbol{\sigma}_{\mathsf{C}}\left(\mathbf{x}+\mathbf{r}|\,\mathbf{x},N-1\right)\cdot\mathbf{n}}\right)^{\mathsf{T}} -$$

$$\left. -\frac{2}{3}\mathbf{I}\left(\nabla\cdot\overline{\int\limits_{|\mathbf{r}|=\mathsf{a}} dS_{\mathsf{r}}\boldsymbol{\sigma}_{\mathsf{C}}\left(\mathbf{x}+\mathbf{r}|\,\mathbf{x},N-1\right)\cdot\mathbf{n}}\right)\right] +$$

$$+\epsilon_{\mathsf{ijk}}\frac{1}{2}\left[\overline{n\int\limits_{|\mathbf{r}|=\mathsf{a}} dS_{\mathsf{r}}\left(\boldsymbol{\sigma}_{\mathsf{C}}\cdot\mathbf{n}\right)\times\mathbf{r}} - \frac{1}{2}\nabla\cdot\left(\overline{n\int\limits_{|\mathbf{r}|=\mathsf{a}} dS_{\mathsf{r}}\mathbf{r}\left(\left(\boldsymbol{\sigma}_{\mathsf{C}}\cdot\mathbf{n}\right)\times\mathbf{r}\right)}\right) + \right.$$

$$\left. + \frac{1}{2}\nabla\nabla\colon \left(\overline{n\int\limits_{|\mathbf{r}|=\mathsf{a}} dS_{\mathsf{r}}\mathbf{r}\mathbf{r}\left(\left(\boldsymbol{\sigma}_{\mathsf{C}}\cdot\mathbf{n}\right)\times\mathbf{r}\right)}\right)\bigg]\bigg)\bigg]\right] +$$

$$+\rho_{\mathsf{D}}\nabla\cdot\left(\beta_{\mathsf{C}}\left\langle \mathbf{u}_{\mathsf{C}}\right\rangle\left\langle \mathbf{u}_{\mathsf{C}}\right\rangle - \left\langle \mathbf{u}_{\mathsf{C}}\mathbf{u}_{\mathsf{C}}\right\rangle\right) - \beta_{\mathsf{C}}\nabla\psi_{\mathsf{C}} +$$

$$+\frac{a^2}{10}\left[(\nabla n)\times\left(\nabla\times\overline{\int\limits_{|\mathbf{r}|=\mathsf{a}} dS_{\mathsf{r}}\boldsymbol{\sigma}_{\mathsf{C}}\left(\mathbf{x}+\mathbf{r}|\,\mathbf{x},N-1\right)\cdot\mathbf{n}}\right) + \right.$$

$$\left. + n\nabla\left(\nabla\cdot\overline{\int\limits_{|\mathbf{r}|=\mathsf{a}} dS_{\mathsf{r}}\boldsymbol{\sigma}_{\mathsf{C}}\left(\mathbf{x}+\mathbf{r}|\,\mathbf{x},N-1\right)\cdot\mathbf{n}}\right)\right] + \ldots.$$

where all the tensors present as an another challenge to overcome.

It should be noted that these complicated expressions still do not have a term reflecting the velocity of the interface surface to reflect the movement and change of the relative position of the interface surface. Nor do they have a term with interface velocity-production, $\frac{1}{\Delta\Omega}\int\limits_{\partial\mathsf{S}_w} U_{\mathsf{C}\mathsf{j}}U_{\mathsf{C}\mathsf{i}}\cdot\overrightarrow{ds}$.

Further, the right hand side does not have terms reflecting processes ”on” and ”along” the interface surface. These terms are absent (although present in the VAT models) because the author, and many other in the two-phase flow field, do not perform the double averaging of the right hand side tensorial operators. This means that these results (in current "......") and others by Prosperetti and co-authors are the consequences of the perturbation expansion of the stress tensor and will remain so unless the double averaging of it is somehow (maybe not possible) incorporated into the ensemble averaging methodology. It is not surprising that the only method of closure they turn to - is the Direct Numerical Simulation of the initial lower level (scale) homogeneous equations.

Some Specific Comments

On "......" stated - "Our work starts from the premise that today, for the

first time, the detailed information necessary for the development of better models is available from direct numerical simulation." This statement is incorrect - there are other methods in closely related fields that have produced exact numerical results and information to improve heterogeneous modeling. Further, the governing equations developed by Prosperetti are incomplete and have deficiencies.

On "...." - "It should be noted that, prior to this work, no technique existed for

the rigorous calculation of ensemble averages of spatially non-uniform systems," This is incorrect, several other researchers (Whitaker, Quintar, Travkin and others) have developed and used "rigorous calculations" of heterogeneous systems. It is further stated that - "closure laws are derived in the form of functional relations that are, at least formally, problem - independent,".

The general approach to closure proposed by Prosperetti is to declare up-front a model for closure and then to follow the results of numerical simulation to obtain the coeffcients in the closure model. On page "..." it is stated - "The simultaneous consideration of the closure relations as well as to validate the closure itself ...". For example, on page 6 the relationship given for the closure of the stress tensor $\overleftrightarrow{S}$ is

$$\overleftrightarrow{S} = 2\mu_{\mathsf{eff}} E_{\mathsf{m}} + 2\mu_{\Delta} E_{\Delta} + 2\mu_{\nabla} E_{\nabla},$$

where the three effective viscosities are just "postulated". This method might supress the physical and mathematical basics of the underlaying phenomena in favor of model convenience.

On page "...." there is the statement - "In averaged-equations models of the "two-fluid" type, the particles are described in terms of the local volume fraction β_{D}..." This supposition is a simplification of a known situation and leads naturally to the conclusion that - "the very foundation of the two-fluid modeling carried out in the last several decades fails."

On the page "...." - "it is meaningful, for example, to simulate situations in which the imposed forces, torques, velocities, and angular velocities are not equal but are selected from assigned probability distributions." It appears as if Prosperetti wants to assign the major properties of a particulate flow to each particle. This will eliminate the connection between physically justified parameters (meaning those found or simulated with accepted modeling equations) and calculated responses of the particular problem studied. The result could be very different from a solution based on calculated parameters.

It is worth noting that, at present, the VAT, in contrast to ensemble averaging, is developed to a level where allmost all the restrictions cited earlier and enumerated by Buyevich, Drew and Lahey, Prosperetti and in other works can be treated in the VAT through the proper mathematical procedures. VAT has been applied to transport phenomena in heterogeneous media with the following features:

1) multi-scaled media;

2) media with non-linear physical characteristics - including high *Reynolds* numbers and even turbulence;

3) polydisperse morphologies;

4) materials with phase anisotropy;

5) media with non-constant or field dependent phase properties;

6) transient problems;

7) presence of imperfect interface surfaces;

8) presence of the internal (mostly at the interface) physico-chemical phenomena;

It is too bad that many of the efforts in the two-phase flow area have given the method a poor image. Two phase flow is a complicated arena to work in and results were needed and often obtained at the expense of rigor or even good physics. This is an area that needs the close attention of someone like Prof Prosperetti who is a brilliant mathematician. What he has done heretofore is much better than any of his predecessors in this area. His choice of ensemble averaging method is unfortunate because, as noted by Buyevich, there too many hurdles that require simplification to overcome.