Few Remarks on Prosperetti's studies (by V.S.Travkin)

The mathematical development of two-phase flow equations by Professor Prosperetti is much better and more rigorous than that of his predecessors. There still remains, however, the problems associated with ensemble averaging over statistical entities that are interconnected by various processes or phenomena.

When Buyevich and Theofanous ("Ensemble Averaging Technique in the Mechanics of Suspensions", ASME FED - Vol. 243, pp. 41-60, 1997.) were concerned about the obvious mismatch between multiphase transport governing equations obtained by two different theoretical approaches and state "the confusion due to various, seemingly incongruent, forms of the field equations (particularly of the momentum equations)" ... "this is especially troublesome, and we agree. It is not exactly clear what the practical impacts are, but such a confusion goes to the heart of one's educational effort and undermines the very foundation of the field". Their comments are well taken because for 30 or more years everyone in this field has felt free to "create" his own governing equations, and then to solve them. Prosperetti is no different albeit with more rigor. It is not clear what it means when a less than rigorous solution sometimes compares well with experimental or simplified data..

Two methods have been used to treat problems of the type that are of interest to Prosperetti. There is ensemble averaging as exemplified by Prosperetti and volume averaging theory (VAT) as exemplified by Whitaker and his students or Travkin at UCLA. In what follows you will find a detailed discussion of ensemble averaging in the context of the texts along with problems and some comparison with the results of VAT (the equations derived using VAT are exact although often difficult to solve). The past work of Prosperetti will be relied upon to do this.

Disadvantages of Ensemble Averaging Techniques:

In the work by Zhang and Prosperetti (1994), ("Averaged Equations for Inviscid Disperse Two-

Phase Flow", J. Fluid Mech., Vol. 267, pp. 185-219) two assumptions were made; 1) N identical particles; 2)potential fluid flow. These assumptions have obvious restrictions. The paper contains some very nice comparisons with the results obtained by applying VAT averaging theorems. The gradient of continuous phase volume fraction, $\nabla\beta_{\mathsf{c}}$, given by their equ. (2.17) (note that $\langle m\rangle = \beta_{\mathsf{c}}$)

$$\nabla\beta_{\mathsf{c}} = \int\limits_{|\mathsf{x}-\mathsf{y}|=\mathsf{a}} dS_{\mathsf{y}}\mathbf{n}\int\limits_{\infty} d^3wP\left(1;t\right) = \int\limits_{|\mathsf{x}-\mathsf{y}|=\mathsf{a}} dS_{\mathsf{y}}\mathbf{n}n\left(y,t\right), (2.17), \tag{1}$$

where $n\left(y,t\right)$ is local particle number density and $\mathbf{n}$ is the unit normal vector oriented outward from the particles. This relationship is identical to the expression obtained using VAT,

$$\nabla < m >= -\frac{1}{\Delta\Omega}\int\limits_{\partial\mathsf{S}_w} \overrightarrow{ds} \; = \frac{1}{\Delta\Omega}\int\limits_{\partial\mathsf{S}_w} \overrightarrow{ds}_1, \tag{2}$$

see, for example, Whitaker (1984,1994),

Two other theorems developed by Zhang and Prosperetti are also very close. Zhang and Prosperetti's equ. (2.23)

$$\nabla\left(\beta_{\mathsf{c}}\left\langle f_{\mathsf{c}}\right\rangle\right) = \beta_{\mathsf{c}}\left\langle\nabla f_{\mathsf{c}}\right\rangle + \int\limits_{|\mathsf{x}-\mathsf{y}|=\mathsf{a}} \mathbf{n}dS_{\mathsf{y}}\int\limits_{\infty} d^3wP\left(1;t\right)\left\langle f_{\mathsf{c}}\right\rangle_1\left(\mathbf{x},t\mid 1\right), \quad (2.23), \tag{3}$$

has an analog from VAT theory

$$\nabla\left(\left\langle m\right\rangle\widetilde{p}\right) = \left\langle m\right\rangle\left\{\nabla p\right\}_{\mathsf{f}} - \frac{1}{\Delta\Omega}\int\limits_{\partial\mathsf{S}_w} p\overrightarrow{ds} = \left\langle m\right\rangle\left\{\nabla p\right\}_{\mathsf{f}} + \frac{1}{\Delta\Omega}\int\limits_{\partial\mathsf{S}_w} p\overrightarrow{ds}_1, \tag{4}$$

and their equ (2.24)

$$\nabla\left\langle f_{\mathsf{c}}\right\rangle = \left\langle\nabla f_{\mathsf{c}}\right\rangle + \frac{1}{\beta_{\mathsf{c}}}\int\limits_{|\mathsf{x}-\mathsf{y}|=\mathsf{a}} \mathbf{n}dS_{\mathsf{y}}\int\limits_{\infty} d^3wP\left(1;t\right)\left[\left\langle\nabla f_{\mathsf{c}}\right\rangle_1\left(\mathbf{x},t\mid 1\right) - \left\langle\nabla f_{\mathsf{c}}\right\rangle\left(\mathbf{x},t\right)\right], (2.24), \tag{5}$$

is very similar to its VAT counterpart

$$\nabla\widetilde{p} = \left\{\nabla p\right\}_{\mathsf{f}} - \frac{1}{\left\langle m\right\rangle\Delta\Omega}\int\limits_{\partial\mathsf{S}_w} \widehat{p}\overrightarrow{ds} = \left\{\nabla p\right\}_{\mathsf{f}} + \frac{1}{\Delta\Omega_{\mathsf{f}}}\int\limits_{\partial\mathsf{S}_w} \widehat{p}\overrightarrow{ds}_1. \tag{6}$$

A number of simplifying, and sometimes restrictive, assumptions were made to achieve closure of their derived equations. On page p. 195, it is stated that - "equations derived in the preceding sections contain several terms involving integration over spheres with a radius equal to the particle radius a". They use the dilute limit of small particles to "close" the momentum equations. They also considered the case of "finite volume fractions for the linear problem" formulate the problem equations for inviscid and non-interactive (no convection)(see eqs. (6.1), (6.2)). For this linear case, they compared their work with Sangani et al. (1991) following the further assumption of a "locally uniform pressure gradient" $\mathbf{G}(t)$, a circumstance of only limited academic interest (p. 201, also see Landau and Lifshitz, 1959).

On p. 203 they describe the numerical algorithm they used for a linear problem without collision forces. The ensemble averaging was done as indicated by the statement: "we first calculate volume averages over the fundamental cell and then average these values over the different realizations. It is the result of this combined average that we identify with the ensemble average $\langle f \rangle$ used in the previous sections". As long as they only consider spatial uniformity , meaning homogeneous spatial and statistical distributions, they are simply performing volume averaging.

They write (see p. 203, eq. (8.2)) that "The volume-averaged fluid acceleration $\widetilde{\dot{\mathbf{u}}}_{\mathsf{c}}$ is obtained directly from a knowledge of $\widetilde{\dot{\mathbf{w}}}$ (also volume averaged!) " using equation

$$\rho_{\mathsf{c}} \widetilde{\dot{\mathbf{u}}}_{\mathsf{c}} = \frac{1}{\Delta\Omega_{\mathsf{f}}} \int\limits_{\Delta\Omega_f} (-\nabla p_{\mathsf{c}})\, d\omega,$$

whose right hand side is the intrinsic phase average of the pressure gradient and equal to

$$\rho_{\mathsf{c}} \widetilde{\dot{\mathbf{u}}}_{\mathsf{c}} = -\nabla \langle p_{\mathsf{c}} \rangle - \frac{1}{\Delta\Omega_{\mathsf{f}}} \int\limits_{\partial \mathsf{S}_w} p \vec{ds},$$

is closed by Zhang and Prosperetti as

$$\rho_{\mathsf{c}} \widetilde{\dot{\mathbf{u}}}_{\mathsf{c}} = \frac{1}{\beta_{\mathsf{c}}} \mathbf{G}(t) - \frac{\beta_{\mathsf{D}}}{\beta_{\mathsf{c}}} \rho_{\mathsf{D}} \widetilde{\dot{\mathbf{w}}}, \tag{7}$$

which is the result obtained by VAT applied to an inviscid linear flow equation with constant volume fraction -or porosity $\langle m \rangle$.

Analysis of work by Prosperetti and co-authors in 1994-97 publications.

A few concluding remarks can be made about the "potential equations" stage of Prosperetti's studies. His work presents the most correct description of ensemble averaging methodology. Some of his equations compared well with VAT equations and some are exact. There are a number of severe restrictions to the works of Prosperetti and co-authors that should be mentioned:

1) N identical particles, (in their bubbly flow modeling they treat a monodisperse array of particles and .

2) the fluid flow is potential flow.

Their particle motion equation neglects :

3) full effect of $\nabla \langle p_{\mathsf{c}} \rangle$ - pressure gradient in the fluid,

4) F_{ac} - additional force of relative acceleration of fluid around the particle;

5) F_{B} - Basset's term influensing the nonsteadyness of fluid flow around the particle (hereditary force);

6) Stokes law influence on particle velocity;

7) Effective buoyancy force and

8) the momentum equation is not applicable at large Re_{p} (particle Reynolds number).

Most of the assumptions and criticisms found the work by Buyevich assumptions are applicable to the develoments by Zhang and Prosperetti, as noted in the following:

9) the drag forces exerted by the ambient fluid are linear in the relative fluid velocity (applicable only fine particle suspensions),

10) "all surface tension effects are ignored, so that stresses have no discontinuity at the interface";

11) overlook possible contributions to the effective stresses acting in a mean suspension flow

due to fluctuations,

12) "spheres to be free from imbedded dipole moments, so that there is no dipole interaction of the spheres between themselves and with a corresponding external field";

13) the relevant and interactive fields of particle positions, velocities, accelerations and angular velocities are taken as independent variables, so that the "strong friction" approximation undertaken to simplify the development implies that only "the positions vectors alone, ... are quite sufficient to characterize possible configurations of the particulate ensemble"; and

14) The closure methods were developed for dilute cases (important to some applications).

Comparisons were made with Drew's work but only for the exceptional dilute case.Substantial comparison with work by Wallis (1991a,b) was done. Criticism was made of Wallis (1991a,b) for "his use of area and volume", as opposed to ensemble averaging. Although all these averaging techniques coincide for homogeneous systems, care is needed in interpreting spatial averages for dense, non-homogeneous mixtures whereas ensemble averages are always well defined. A careful comparison made case by case, situatiuon by situation, feature by feature, equation by equation shows the supremacy of VAT in completenes and connections to limiting situations and practical needs.

To comment on "....", we need to outline the differences with his work done during the 1994-97 (potential flow) period and what is found in the text of the "....". We focus on one particular important publication where the most important features of recent advancements by Prosperetti are shown. This is the paper by Marchioro, Tanksley, and Prosperetti, (1999)("Mixture Pressure and Stress in Dispersive Two-Phase Flow," *Intern. J. Multiphase Flow*, **25**, pp. 1395-1429).

On page p. 1396, it is stated "However, when the viscosity is large enough, the behavior of the drops would be indistinguishable from that of rigid particles and yet, although the average flow would be exactly the same in the two cases. the concept of "pressure" inside a rigid particle would be devoid of physical meaning." This is an engineering approximation and detracts from

the claimed rigor of the "....."

On p. 1396 it is noted that -"Several authors avoid the introduction of disperse-phase pressure and replace it by an "interfacial pressure", related to the mean continuous-phase pressure in the neighborhood of the particles (see, e.g., Anderson and Jackson, 1967; Ishii, 1975; Drew, 1983; Prosperetti and Jones, 1984; Arnold et al., 1989)." We will now look at the mathematical construction and see how they treat this aspect of the problem.

The equation for the continuous state is

$$\rho_{\mathsf{C}} \left[\frac{\partial \mathbf{u}_{\mathsf{C}}}{\partial t} + \nabla \cdot (\mathbf{u}_{\mathsf{C}} \mathbf{u}_{\mathsf{C}}) \right] = \nabla \cdot \boldsymbol{\sigma}_{\mathsf{C}} - \nabla \psi_{\mathsf{C}}$$

where σ_{C} is the stress, and ψ_{C} is the potential of the body force. In the case where the potential is gravitational, they wrote $\psi_{\mathsf{C}} = -\rho_{\mathsf{C}} g \cdot x$, instead of $\nabla \psi_{\mathsf{C}} = -\rho_{\mathsf{C}} g \cdot x$.Marchioro et al. apply averaging separately to each side of the equation. They write

$$\mathbf{I}_{\mathsf{C}} = \beta_{\mathsf{C}} \langle \nabla \cdot \boldsymbol{\sigma}_{\mathsf{C}} \rangle - \beta_{\mathsf{C}} \nabla \psi_{\mathsf{C}}, \quad (3),$$

where the angle brackets denote the phase-ensemble average and β_{C} is the volume fraction of the continuous phase, which they note can be inhomogeneous in space. It is not clear why the authors did not average the potential term $\nabla \psi_{\mathsf{C}}$ correctly. It is easily shown that the gradient operator is not averaged just by multipliyng by β_{C}.

In their notations, the right hand side of this equation should be

$$\langle \nabla \psi_{\mathsf{C}} \rangle_{\mathsf{C}} = \nabla \left(\beta_{\mathsf{C}} \langle \psi_{\mathsf{C}} \rangle \right) + \int_{|\mathbf{x}-\mathbf{y}|=\mathsf{a}} dS_{\mathsf{y}} P(\mathbf{y}) \langle \psi_{\mathsf{C}} (\mathbf{x}|\, \mathbf{y}, t) \rangle_1 \cdot \mathbf{n}_{\mathsf{y}}.$$

The left part becomes

$$\mathbf{I}_{\mathsf{C}} = \rho_{\mathsf{C}} \left[\frac{\partial}{\partial t} \left(\beta_{\mathsf{C}} \langle \mathbf{u}_{\mathsf{C}} \rangle \right) + \nabla \cdot \left(\beta_{\mathsf{C}} \langle \mathbf{u}_{\mathsf{C}} \mathbf{u}_{\mathsf{C}} \rangle \right) \right], \quad (4)$$

Note, in equation (126) - page 1417, the space spacial property of the volume fraction β_{C} is taken to be constant and moved from within the averaging operator $\langle \rangle$. This is incorrect.

The term $\nabla \cdot (\beta_{\mathsf{C}} \langle \mathbf{u}_{\mathsf{C}} \mathbf{u}_{\mathsf{C}} \rangle)$ should look like (in VAT notations)

$$\langle \nabla \cdot (\beta_{\mathsf{C}} \mathbf{u}_{\mathsf{C}} \mathbf{u}_{\mathsf{C}}) \rangle_{\mathsf{C}} =$$

$$\begin{aligned}
&= \nabla < \widetilde{\mathbf{u}}_{\mathsf{C}} \widetilde{\mathbf{u}}_{\mathsf{C}} + \widehat{\mathbf{u}}_{\mathsf{C}} \widehat{\mathbf{u}}_{\mathsf{C}} >_{\mathsf{C}} + \frac{1}{\Delta\Omega} \int\limits_{\partial \mathsf{S}_w} U_{\mathsf{C}\mathsf{j}} U_{\mathsf{C}\mathsf{i}} \cdot \vec{ds} = \\
&= \nabla < \widetilde{\mathbf{u}}_{\mathsf{C}} \widetilde{\mathbf{u}}_{\mathsf{C}} + \widehat{\mathbf{u}}_{\mathsf{C}} \widehat{\mathbf{u}}_{\mathsf{C}} >_{\mathsf{C}} + \frac{1}{\Delta\Omega} \int\limits_{\partial \mathsf{S}_w} U_{\mathsf{C}\mathsf{j}} U_{\mathsf{C}\mathsf{i}} \cdot \vec{ds} = \qquad (8) \\
&= \nabla < \widetilde{\mathbf{u}}_{\mathsf{C}} \widetilde{\mathbf{u}}_{\mathsf{C}} >_{\mathsf{C}} + \nabla < \widehat{\mathbf{u}}_{\mathsf{C}} \widehat{\mathbf{u}}_{\mathsf{C}} >_{\mathsf{C}} + \frac{1}{\Delta\Omega} \int\limits_{\partial \mathsf{S}_w} U_{\mathsf{C}\mathsf{j}} U_{\mathsf{C}\mathsf{i}} \cdot \vec{ds} = \qquad (9)
\end{aligned}$$

$$\begin{aligned}
&= \nabla \left(\beta_{\mathsf{C}} \left\{ \widetilde{\mathbf{u}}_{\mathsf{C}} \widetilde{\mathbf{u}}_{\mathsf{C}} \right\}_{\mathsf{C}}\right) + \nabla < \widehat{\mathbf{u}}_{\mathsf{C}} \widehat{\mathbf{u}}_{\mathsf{C}} >_{\mathsf{C}} + \frac{1}{\Delta\Omega} \int\limits_{\partial \mathsf{S}_w} U_{\mathsf{C}\mathsf{j}} U_{\mathsf{C}\mathsf{i}} \cdot \vec{ds} = \\
&= \nabla \left(\beta_{\mathsf{C}} \left\{ \widetilde{\mathbf{u}}_{\mathsf{C}} \widetilde{\mathbf{u}}_{\mathsf{C}} \right\}_{C}\right) + \nabla \left(\beta_{\mathsf{C}} \left\{ \widehat{\mathbf{u}}_{\mathsf{C}} \widehat{\mathbf{u}}_{\mathsf{C}} \right\}_{\mathsf{C}}\right) + \frac{1}{\Delta\Omega} \int\limits_{\partial \mathsf{S}_w} U_{\mathsf{C}\mathsf{j}} U_{\mathsf{C}\mathsf{i}} \cdot \vec{ds},
\end{aligned}$$

The expression derived by Prosperetti does not have the term with the interface velocity-production $\frac{1}{\Delta\Omega} \int\limits_{\partial \mathsf{S}_w} U_{\mathsf{C}\mathsf{j}} U_{\mathsf{C}\mathsf{i}} \cdot \vec{ds}$. Prosperetti and colleagues do not have this term in their earlier papers of 94-97.

They explain the averaging of gradient terms as

$$\begin{aligned}
\beta_{\mathsf{C}}(\mathbf{x}) \langle \nabla \cdot \boldsymbol{\sigma}_{\mathsf{C}}(\mathbf{x},t) \rangle &= \nabla \cdot (\beta_{\mathsf{C}}(\mathbf{x}) \langle \boldsymbol{\sigma}_{\mathsf{C}}(\mathbf{x},t) \rangle) + \\
&\quad + \int\limits_{|\mathbf{x}-\mathbf{y}|=\mathsf{a}} dS_{\mathsf{y}} P(\mathbf{y}) \langle \boldsymbol{\sigma}_{\mathsf{C}}(\mathbf{x}|\,\mathbf{y},t) \rangle_1 \cdot \mathbf{n}_{\mathbf{y}}, \qquad (10)
\end{aligned}$$

where $P(\mathbf{y})$ is the single-particle probability density defined in Eq. (A7) and $\langle \boldsymbol{\sigma}_{\mathsf{C}}(\mathbf{x}|\,\mathbf{y},t) \rangle_1$ is the stress at $\mathbf{x}$ averaged conditionally (see the definition (A8)) to the presence of a particle with center at $\mathbf{y}$.Here we need to return, for a more precise description, to their definitions of the averaged variables, functions and operators.

First, the phase average as it appeared in their 1994 paper,

$$\langle f_{\mathsf{C,D}}\rangle(\mathbf{x},t)=\frac{1}{N!\beta_{\mathsf{C,D}}}\int_{\infty}d\theta^{\mathsf{N}}P(N;t)\,\chi_{\mathsf{C,D}}(\mathbf{x};N)\;\;f_{\mathsf{C,D}}(\mathbf{x},t;N),\quad (A6), \tag{11}$$

with the one particle probability distribution $P(\mathbf{y},\mathbf{w})$ defined as

$$P(1)\equiv P(\mathbf{y},\mathbf{w})=\frac{1}{(N-1)!}\int_{\infty}d\theta^{(\mathsf{N}-1)}P(N),\quad (A7)$$

and the one particle conditional average given by

$$\beta^{1}_{\mathsf{C}}\langle f_{\mathsf{C}}\rangle_{1}(\mathbf{x},t|\,\mathbf{y},\mathbf{w})=\frac{1}{(N-1)!}\int_{\infty}d\theta^{(\mathsf{N}-1)}\chi_{\mathsf{C}}(\mathbf{x};N)\,f_{\mathsf{C}}(\mathbf{x},t;N)\,P(N-1|\,1),\quad (A8),$$

where the conditional probability $P(N-1|\,1)$ is defined by $P(N)=P(1)\,P(N-1|\,1)$.

They use this averaging expression (page 1398) for the first (and most important) right hand side term with a perturbation expansion of the function $\beta_{\mathsf{C}}(\mathbf{x})\langle\nabla\cdot\boldsymbol{\sigma}_{\mathsf{C}}(\mathbf{x},t)\rangle$ in the form

$$\begin{aligned}\beta_{\mathsf{C}}(\mathbf{x})\langle\nabla\cdot\boldsymbol{\sigma}_{\mathsf{C}}(\mathbf{x},t)\rangle &= \nabla\cdot(\beta_{\mathsf{C}}(\mathbf{x})\langle\boldsymbol{\sigma}_{\mathsf{C}}(\mathbf{x},t)\rangle)-\\ &\quad -nA[\boldsymbol{\sigma}_{\mathsf{C}}]+\nabla\cdot(\beta_{\mathsf{D}}(\mathbf{x})\,L[\boldsymbol{\sigma}_{\mathsf{C}}]),\quad (6),\end{aligned} \tag{12}$$

where n is the particle number density defined in Eq. (A5) and there results

$$\begin{aligned}\beta_{\mathsf{D}}(\mathbf{x})\,L[\boldsymbol{\sigma}_{\mathsf{C}}] &= nT[\boldsymbol{\sigma}_{\mathsf{C}}]+\\ &\quad +\nabla\cdot\{nI[\boldsymbol{\sigma}_{\mathsf{C}}]+\nabla\cdot[nR[\boldsymbol{\sigma}_{\mathsf{C}}]+....]\},\quad (7),\end{aligned} \tag{13}$$

with

$$A[\boldsymbol{\sigma}_{\mathsf{C}}](\mathbf{x})=\overline{\int_{|\mathbf{r}|=\mathsf{a}}dS_{\mathsf{r}}\boldsymbol{\sigma}_{\mathsf{C}}(\mathbf{x}+\mathbf{r}|\,\mathbf{x},N-1)\cdot\mathbf{n}},$$

On page 1399 they state "Here the overline denotes the particle average defined in Eq. (A9)), i.e., the ensemble average over all the configurations such that one of the particles has center at $\mathbf{x}$; the

integration is over the surface of that particle. The terms neglected in Eq. (7) are of higher order in a/L."To complete these equations, the following terms are given by

$$T\left[\boldsymbol{\sigma}_{\mathsf{C}}\right](\mathbf{x}) = a\overline{\int_{|\mathbf{r}|=\mathsf{a}} dS_{\mathsf{r}}\mathbf{n}\left[\boldsymbol{\sigma}_{\mathsf{C}}\left(\mathbf{x}+\mathbf{r}\right|\mathbf{x},N-1)\cdot\mathbf{n}\right]},$$

$$I\left[\boldsymbol{\sigma}_{\mathsf{C}}\right](\mathbf{x}) = -\frac{1}{2}a^2\overline{\int_{|\mathbf{r}|=\mathsf{a}} dS_{\mathsf{r}}\mathbf{nn}\left[\boldsymbol{\sigma}_{\mathsf{C}}\left(\mathbf{x}+\mathbf{r}\right|\mathbf{x},N-1)\cdot\mathbf{n}\right]},$$

$$R\left[\boldsymbol{\sigma}_{\mathsf{C}}\right](\mathbf{x}) = \frac{1}{6}a^3\overline{\int_{|\mathbf{r}|=\mathsf{a}} dS_{\mathsf{r}}\mathbf{nnn}\left[\boldsymbol{\sigma}_{\mathsf{C}}\left(\mathbf{x}+\mathbf{r}\right|\mathbf{x},N-1)\cdot\mathbf{n}\right]}.$$

With this and creating the intermediate steps in Prosperetti's development, the equality defined by Prosperetti (equ. 10) requires that the second term on the right hand side must be

$$\int_{|\mathbf{x}-\mathbf{y}|=\mathsf{a}} dS_{\mathsf{y}}P\left(\mathbf{y}\right)\left\langle\boldsymbol{\sigma}_{\mathsf{C}}\left(\mathbf{x}\right|\mathbf{y},t)\right\rangle_1\cdot\mathbf{n}_{\mathbf{y}} = nA\left[\boldsymbol{\sigma}_{\mathsf{C}}\right]-\nabla\cdot\left(\beta_{\mathsf{D}}\left(\mathbf{x}\right)L\left[\boldsymbol{\sigma}_{\mathsf{C}}\right]\right) =$$

$$\begin{aligned}
&= n\overline{\int_{|\mathbf{r}|=\mathsf{a}} dS_{\mathsf{r}}\boldsymbol{\sigma}_{\mathsf{C}}\left(\mathbf{x}+\mathbf{r}\right|\mathbf{x},N-1)\cdot\mathbf{n}}- \\
&\quad -\nabla\cdot\left(na\overline{\int_{|\mathbf{r}|=\mathsf{a}} dS_{\mathsf{r}}\mathbf{n}\left[\boldsymbol{\sigma}_{\mathsf{C}}\left(\mathbf{x}+\mathbf{r}\right|\mathbf{x},N-1)\cdot\mathbf{n}\right]}\right)+ \\
&+\nabla\nabla\cdot\left(n\frac{1}{2}a^2\overline{\int_{|\mathbf{r}|=\mathsf{a}} dS_{\mathsf{r}}\mathbf{nn}\left[\boldsymbol{\sigma}_{\mathsf{C}}\left(\mathbf{x}+\mathbf{r}\right|\mathbf{x},N-1)\cdot\mathbf{n}\right]}\right)- \\
&-\nabla\nabla\nabla\cdot\left(n\frac{1}{6}a^3\overline{\int_{|\mathbf{r}|=\mathsf{a}} dS_{\mathsf{r}}\mathbf{nnn}\left[\boldsymbol{\sigma}_{\mathsf{C}}\left(\mathbf{x}+\mathbf{r}\right|\mathbf{x},N-1)\cdot\mathbf{n}\right]}\right)-..... \qquad (14)
\end{aligned}$$

This is a very difficult looking expression.

The finished form of this equation becomes

$$\mathbf{I}_{\mathsf{C}} = \nabla\cdot\left(\beta_{\mathsf{C}}\left(\mathbf{x}\right)\left\langle\boldsymbol{\sigma}_{\mathsf{C}}\left(\mathbf{x},t\right)\right\rangle+\beta_{\mathsf{D}}\left(\mathbf{x}\right)L\left[\boldsymbol{\sigma}_{\mathsf{C}}\right]\right)-nA\left[\boldsymbol{\sigma}_{\mathsf{C}}\right]-\beta_{\mathsf{C}}\nabla\psi_{\mathsf{C}}, \quad (13).$$

The equation for the disperse phase after the analogous averaging

$$\mathbf{I}_{\mathsf{D}} = \beta_{\mathsf{D}} \left\langle \nabla \cdot \boldsymbol{\sigma}_{\mathsf{D}} \right\rangle - \beta_{\mathsf{D}} \nabla \psi_{\mathsf{D}}, \quad (15),$$

and then after expansion into a Taylor series, the gradient $\beta_{\mathsf{D}} \left\langle \nabla \cdot \boldsymbol{\sigma}_{\mathsf{D}} \right\rangle$ on the right hand side becomes

$$\begin{aligned} \mathbf{I}_{\mathsf{D}} &= nA_{\mathsf{D}} \left[\boldsymbol{\sigma}_{\mathsf{D}} \right] + \nabla \cdot \overleftrightarrow{\boldsymbol{\Sigma}}_{\mathsf{a}} - \\ &\quad -\beta_{\mathsf{D}} \nabla \psi_{\mathsf{D}}, \quad (16), \end{aligned}$$

where the stress tensor $\overleftrightarrow{\boldsymbol{\Sigma}}_{\mathsf{a}}$ is "conceptually similar to L"; its explicit expression is just another series expansion (see equ. (19) in Prosperetti) and is not provided here. Note, however, that the expression includes the factor $\left\langle \nabla_{\mathsf{r}} \cdot \boldsymbol{\sigma}_{\mathsf{D}} \right\rangle$, meaning the the average of the stress tensor is still present in the mathematical expressions. Evaluation of this term is not an easy task.

Then authors lamped the both equations together and got the equation they often refer later

$$\mathbf{I}_{\mathsf{C}} + \mathbf{I}_{\mathsf{D}} = \nabla \cdot \left(\beta_{\mathsf{C}} \left\langle \boldsymbol{\sigma}_{\mathsf{C}} \right\rangle + \beta_{\mathsf{D}} L \left[\boldsymbol{\sigma}_{\mathsf{C}} \right] + \overleftrightarrow{\boldsymbol{\Sigma}}_{\mathsf{a}} \right) - \beta_{\mathsf{C}} \nabla \psi_{\mathsf{C}} - \beta_{\mathsf{D}} \nabla \psi_{\mathsf{D}}, \quad (20).$$

VAT theory yields a much simpler equation of the form, in VAT notation, for the stress term,

$$< \frac{\partial}{\partial x_{\mathsf{j}}} (2\mu S) >_{\mathsf{f}} = < \nabla \cdot (2\mu S) >_{\mathsf{f}} = \nabla \cdot (< 2\mu S >_{\mathsf{f}}) +$$

$$+ \frac{1}{\Delta\Omega} \int\limits_{\partial \mathsf{S}_w} 2\mu S \cdot \vec{ds} =$$

$$\nabla \cdot 2 \left[\langle m \rangle \widetilde{\mu} \widetilde{S} + \langle m \rangle \left\{ \widehat{\mu} \widehat{S} \right\}_{\mathsf{f}} \right] + \frac{2}{\Delta\Omega} \int\limits_{\partial \mathsf{S}_w} \mu S \cdot \vec{ds}, \quad (15)$$

where S is the stress tensor. The second and third terms could also be represented by a series expansion as was done by Marchioro et al. (1999). This breif remark should be kept in mind when considering the usefulness of the ensemble averaging.