The current text is not intended to provide the comprehensive overview neither of the ensemble averaging techniques and their peculiarities nor the resulting mathematical governing equations of transport in dispersed media. Nevertheless, as long as there is the concern related to analysis of overall situation with the science of heterogeneous media transport mathematical modeling, few of the works need to be analyzed with the purpose to describe the methods used in ensemble averaging approach (and in few works - volume averaging technique) and their resulting equations which, unfortunately are different in many instances.

Sufficiently important in this situation are simplifying assumptions.

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0.0.1 J.A.DeSanto, ed., Mathematical and Numerical Aspects of Wave Propagation, SIAM, Philadelphia, 1998.

There is the gap of ~15 years between the advancements in the fluid mechanics and thermal physics and wave propagation modeling sciences. The latter is behind, and does not know about that.

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1) Among numerous works on dispersed medium modeling are studies by Lahey, Drew and co-authors.

0.0.2 +Lahey, R.T., (1996), "A CFD Analysis of Multidimensional

Two-Phase Flow and Heat Transfer Phenomena",

in *Process, Enhanced, and Multiphase Heat Transfer*, eds. R.M.Manglik and A.D.Kraus, Begel House, New York, pp. 431-441.

In this paper grouped and summarized the developments and acheivements that were done in the area of multiphase flow transport modeling. As author asserts "It will be shown that bubbly vapor/liquid and solid/fluid slurry flows can now be predicted using essentially the same two-fluid model. Indeed, it appears that this approach may have the ability to completely unify the field of two-phase flow".

Starting from formulation of the multiphase ensemble averaged equations (almost completely fortotten the models and studies done with volume averaging method - see works with Drew et al.) author comes to the equations of conservation of mass, momentum, energy, turbulent kinetic energy, and turbulence dissipation rate written in some general form based on one type generic local equations in each of the phase

$$\frac{\partial\left(\rho\vec{\mathbf{\Psi}}\right)}{\partial t}+\nabla\cdot\left(\rho\vec{\mathbf{\Psi}}\vec{\mathbf{V}}\right)=\nabla\cdot\underline{\underline{\mathbf{J}}}+\rho f, \tag{1}$$

with corresponding jump condition between phases k and l

$$\left[\left(\rho\vec{\mathbf{\Psi}}\left(\vec{\mathbf{V}}-\vec{\mathbf{V}}_i\right)+\underline{\underline{\mathbf{J}}}\right)\cdot\vec{\mathbf{n}}\right]_{kl}=\mathbf{M}_i. \tag{2}$$

Despite of claim to perform the ensemble averaging over the local fields equations this procedure done in a very simplified incorrect way neglecting many issues (see, for comparison, works by Buyevich on ensemble averaging approach) and finally the equations are getting with no more justified appearence plus with then addition of some physically needed terms reflecting needed to be included known phenomena of phase interactions. Thus, basic generic equation is shown in this form

$$\frac{\partial \left(\overline{X_k \rho \vec{\mathbf{\Psi}}}\right)}{\partial t} + \nabla \cdot \left(\overline{X_k \rho \vec{\mathbf{\Psi}} \vec{\mathbf{V}}}\right) = \nabla \cdot \left(\overline{X_k \underset{=}{\mathbf{J}}}\right) - \overline{\underset{=}{\mathbf{J}} \cdot \nabla X_k} + \overline{X_k \rho f} \ . \tag{3}$$

The equation of momentum conservation is not much different then in their previous works when they claim was used volume averaging method for equation derivation. Exactly in this work (Lahey, 1996) ensemble averaged momentum equation appeared as (equation (10), p. 432)

$$\frac{\partial \left(\alpha_k \overline{\rho_k} \overline{\vec{\mathbf{V}}}_k\right)}{\partial t} + \nabla \cdot \left(\alpha_k \overline{\rho_k} \overline{\vec{\mathbf{V}}}_k \overline{\vec{\mathbf{V}}}_k\right) = -\nabla \left(\alpha_k \overline{p_k}\right) +$$

$$+\nabla \cdot \left(\alpha_k \left[\underset{=k}{\overline{\tau}} + \underset{=k}{\overline{\tau}}^{\mathrm{Re}}\right]\right) + \overrightarrow{M}_k + \overline{\rho \vec{\mathbf{V}} \left(\vec{\mathbf{V}} - \vec{\mathbf{V}}_i\right) \cdot \nabla X_k} + \alpha_k \overline{\rho_k} \overline{\vec{g}}_k \ , \tag{4}$$

while in work by Lahey and Lopez de Bertodano (1991) referring to their basic fundamental work on volume averaging method - Lahey and Drew (1988), the same equation looks like (Lahey and Lopez de Bertodano 1991, p. 195, eq. (2))

$$\alpha_k \rho_k \left(\frac{\partial \left(\overline{\vec{\mathbf{V}}}_k\right)}{\partial t} + \nabla \cdot \left(\overline{\vec{\mathbf{V}}}_k \overline{\vec{\mathbf{V}}}_k\right)\right) = -\alpha_k \nabla p_k +$$

$$+\nabla \cdot \left(\alpha_k \left[\mu_k \nabla \overline{\vec{\mathbf{V}}}_k - \rho_k \left(\overline{\vec{u}'_k \vec{u}'_k}\right)\right]\right) +$$

$$+\overrightarrow{M}_{ik} - \overrightarrow{M}_{wk} - \underset{=k_i}{\overline{\tau}} \cdot \nabla \alpha_k + \left(p_{k_i} - p_k\right) \nabla \alpha_k - \alpha_k \rho_k \vec{g} \ , \tag{5}$$

while in the methodical and fundamental work by Lahey and Drew (1988) the same equation is written again differently, using fourth order correlation tensor and incorrectly defined vectorial specific area function (see analysis of that work below).

Further basic affort devoted to modeling of those many additional terms through the separate scale models and coefficient introduction and closure of those micromodels. For that purposes used, for example, the inviscid flow theory for the continuous phase, and "cell-model averaging techniques (Park, 1992; Lopez de Bertodano, 1992)".

The cell-model averaging technique suggested appeared as the kind of effective medium approach allowing to consider separate one cell problem to close the coefficient problems in averaged equations.

One of advantages acclaimed by Lahey (1997) is that "It should be stressed that a significant difference between this two-phase turbulence model, and others, is that the turbulence structure of both phases has been explicitly modelled." Still the complexity of the theory can be illustrated by the fact that it incudes 52 equations.

The comparison of the numerical similation performed using PHOENICS CFD code with experiments showed a great deal of agreement between both. As can be seen from the rigorous theoretical point of view the modeling approach which has been advansing by this group of researchers is flawed in many issues, still demonstrating how much troubleshooting capabilities entailed by taking proper care on tuning and adjusting significant details of the modeling approach.

Despite the bright outcome shown and conclusion stated in the paper - "It appears that fundamental progress can be made, and there **do not appear to be any significant obstacles to prevent us from realizing the goal** (outlined by us) of developing a reliable multidimensional two-fluid model for industrial application", it seems inevitable that the existing fundamental flaws will squeeze throughout themselves in the situation unusual for current modeling adjustments.

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0.0.3 +Lahey, R.T., Jr. and Drew, D.A., (1988), "The three-dimensional

time and volume averaged conservation equations of two-phase flow", in *Advances in Nuclear Science and Technology*, Lewins and Becker, eds., Vol. 20, pp. 1-69.

The most detailed and technical derivation of the two-phase governing equations based on the volume averaging methodology. Among many flaws the main is that the authors tried to develop their own volume averaging technique not knowing or ignoring that much of the theory already been developed in works by Whitaker and Grey - laminar and linear equations, and by Travkin et al. - nolinear and turbulent transport.

From that introduction it is been understood why authors were trapped with errors and inconsistencies.

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Few of issues they done incorrectly or insufficiently:

1) they do not have average of the type $\langle f \rangle_f$, they have only $\{f\}_f$ in (21);

2) they introduced interfacially averaged pressure $\langle p_k \rangle_i$ and stress $\left\langle \underset{=k}{\tau} \right\rangle_i$ as separate variables in (39a), (39b);

3) they introduced interfacial pressure and stress fluctuations in (40a), (40b), but not volume fluctuations;

4) they are dead wrong introducing in (42) the vectorial interfacial area through the vectorial surface integral (which in case of closed regular or even purterbuded regular 3D suface will be zero or almost zero and don't bear any useful information while
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taken in this form)

$$\overrightarrow{S}'''_{k_i} = \frac{1}{V} \iint\limits_{a_i(x,t)} \vec{\mathbf{n}} dS,$$

- because they did not know how to use averaging theorems and their consequencies;

5) consequently, they are wrong in what they write in (48) modeling terms with the "vectorial interfacial area $\overrightarrow{S}'''_{k_i}$" and interfacial fluctuations" following (42) - as long as (48) is actually containing pressure and stress integrals over the interface - good subject for closure, but in this situation would give zero's or close to zero values;

6) they are wrong when mixing interfacial pressure integrals with integrated shear stress in (50) (page 15) - meaning that the surface averaged interfacial pressure can bear some responsibility for the integrated shear effects?

7) they do have method of approaching to the nonlinearity problem with the convection term in the left hand side - "we must write the spatial average of the product of the dependent variables in terms of the product of the spatially averaged variables. It is convenient to accomplish this through the use of defined (new) variables and correlation coefficients" - p. 28.

8) as the consequence of this concept they wrote the velocity product from the left side momentum equation convection term through the mass-weighted averaged product (93) using

$$\not< \overline{\vec{\mathbf{V}}_k \vec{\mathbf{V}}_k} \not>^{\rho}_k = \underline{\underline{C}}_k : \not< \overline{\vec{\mathbf{V}}_k} \not>^{\rho}_k \not< \overline{\vec{\mathbf{V}}_k} \not>^{\rho}_k - \frac{\not< \underline{\underline{\boldsymbol{\tau}}}^{Re}_k \not>}{\not< \bar{\rho}_k \not>_k}, \tag{6}$$

which still was changed later (Lahey and Lopez de Bertodano, 1991) on usual Reynolds stress expression - see equation (2) in Lahey and Lopez de Bertodano (1991);

9) The final momentum equation (108) is being written in the form with introduced in (92), (93) (see above) fourth-order correlation tensor $\underline{\underline{C}}_k$ to overcome the difficulty of treating the nonlinear (the only nonlinearity explicitly adopted in the problem) convective term;

10) But later they silently corrected equation (108) to equation (2) in Lahey and Lopez de Bertodano (1991), regarding

$$\overrightarrow{S}'''_{k_i} \Rightarrow \nabla \alpha_k,$$

then the momentum equation looks like

$$\alpha_k\rho_k\left(\frac{\partial\left(\overrightarrow{\mathbf{V}}_k\right)}{\partial t}+\nabla\cdot\left(\overrightarrow{\mathbf{V}}_k\overrightarrow{\mathbf{V}}_k\right)\right)=-\alpha_k\nabla p_k+$$

$$+\nabla\cdot\left(\alpha_k\left[\mu_k\nabla\overrightarrow{\mathbf{V}}_k-\rho_k\left(\overline{\overrightarrow{u}'_k\overrightarrow{u}'_k}\right)\right]\right)+$$

$$+\overrightarrow{M}_{ik}-\overrightarrow{M}_{wk}-\overline{\underset{=}{\tau_{k_i}}}\cdot\nabla\alpha_k+\left(p_{k_i}-p_k\right)\nabla\alpha_k-\alpha_k\rho_k\overrightarrow{g}\ . \qquad (7)$$

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Still the most important restriction - is that no nonlinearity (apart of convection term) and no heterophase fluctuations admitted to exist.

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0.0.4 +Lahey, R. T. Jr. and Lopez de Bertodano, M., (1991), "The prediction

of phase distribution using two-fluid models", in *Proc. of the ASME/JSME Thermal Engineering Conf.*, Vol. 2, pp. 193-200.

Reffering for the equations development to their work - R.T., Lahey, Jr. and D.A. Drew, (1988), "The three-dimensional....

They have here in momentum equation interfacial force M_{ik} assigned just as is and which further is being modeled separately on the basis of physical considerations. Authors themselves acknowledged that "Closure of such systems is achieved by postulating phase distribution coefficients and interfacial and wall transfer laws which attempt to reintroduce some of the physics which was lost during the averaging process (ie., space/time or ensemble averaging)". Oh, yes, if averaging done incorrectly - that what's happening always.

In phasic momentum equations, as the only they have analyzed, the additional terms appeared which they model using the physical arguments (indirectly) and often intuitively, based on data available and experiments (tuning of the model is done on the basis of "single phase or interface phenomena - averaged bulk multiphase phenomena modeled in the equation").

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0.0.5 Lopez de Bertodano, M., Lee, S-J., Lahey, R. T. Jr., and Drew, D.A., (1990), "The prediction

of two-phase turbulence and phase distribution phenomena using a Reynolds stress model", *J. Fluids Engineering,* Vol. 112, pp. 107-113.

In this paper the Reynolds stress tensor model for continuous phase was used for modeling. Authors used PHOENICS code.

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0.0.6 -Nemat-Nasser, S. and Hori, M., Micromechanics: Overall Properties of Heterogeneous Materials, 2nd edition, Elsevier Science B.V., Amsterdam, 1999.

Extract from "Heterogeneous Electrodynamics ...":

"In accordance with one of the major averaging theorem - theorem of averaging ∇ operator, the WSAM theorem (after Whitaker-Slattery-Anderson-Marle) the averaged operator ∇ becomes

$$\langle \nabla f \rangle_1 = \nabla \langle f \rangle_1 + \frac{1}{\Delta\Omega} \int\limits_{\partial S_{12}} f \vec{ds}_1. \tag{8}$$

Meanwhile, the foundation for averaging made, for example, by Nemat-Nasser and Hori (1999) (and many others) is based on conventional homogeneous Gauss-Ostrogradsky theorem (see pp.59-60), not of its heterogeneous version as by WSAM theorem.

The differentiation theorem for intraphase averaged function is

$$\begin{aligned} \{\nabla f\}_1 &= \nabla \widetilde{f} + \frac{1}{\Delta\Omega_f} \int\limits_{\partial S_w} \widehat{f} \vec{ds}_1, \\ \widehat{f} &= f - \widetilde{f}, \quad f \ \forall \ \Delta\Omega_f, \end{aligned} \tag{9}$$

where ∂S_w is the inner surface in the REV, $\vec{ds}$ is the second-phase, inward-directed differential area in the REV ($\vec{ds} = \vec{n} dS$).

The same kind of operator involving **rot** will result in the following averaging theorem

$$\langle \nabla \times \mathbf{f} \rangle_1 = \nabla \times \langle \mathbf{f} \rangle_1 + \frac{1}{\Delta\Omega} \int\limits_{\partial S_{12}} \vec{ds}_1 \times \mathbf{f}, \tag{10}$$

also as its consequence the another theorem for intraphase average of $\nabla \times \mathbf{f}$

$$\{\nabla \times \mathbf{f}\}_1 = \nabla \times \{\mathbf{f}\}_1 + \frac{1}{\Delta\Omega} \int\limits_{\partial S_{12}} \vec{ds}_1 \times \widehat{\mathbf{f}}. \tag{11}$$

".

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0.0.7 G.Papanicolaou, ed., Wave Propagation in Complex Media, Springer-Verlag, New-York, 1998.

My comments -

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Also see the works by these workers:
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0.0.8 +Zhang, D.Z. and Prosperetti, A., (1994), "Averaged Equations for Inviscid Disperse Two-Phase Flow", *J. Fluid Mech.*, Vol. 267, pp. 185-219.

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D. Z. Zhang and A. Prosperetti, "Averaged Equations for Inviscid Disperse Two-Phase Flow", J. Fluid Mech., 267, 185 (1994).

Zang and Prosperetti derived averaged equations for the motion of equal sized rigid spheres suspended in a potential flow using an equation for the probability distribution. They used the small particle dilute limit approximation to "close" the momentum equations. After approximate resolution of the continuous phase fluctuation tensor Mc and the vector AD(x,t) the fluctuating particle volume flux tensor, MD, they recognized that (p. 199) - "Closure of the system requires an expression for the fluctuating particle volume flux tensor MD .... This missing information cannot be supplied internally by the theory without a specification of the initial conditions imposed on the particle probability distribution". They also considered the case of "finite volume fractions for the linear problem" where the problem equations were formulated for inviscid and unconvectional media. The development by Zhang and Prosperetti13 is a good example of the correct application of ensemble averaging. The equations they derive compare exactly with those derived from rigorous volume averaging theory (VAT) (see Travkin and Catton14).

>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>.

They have assumptions (restrictions): 1) N identical particles.

2) fluid flow is potential !

3)

This paper may have the closest to the VAT based equation forms (may be in some cases coinsiding ? still need more work to be sure on that - 07/30/98).

Few citations will be commented:

From an abstract:

"Averaged equations governing the motion of equal rigid spheres suspended in a potential flow are derived from the equation for the probability distribution. A distinctive feature of this work is the derivation of the disperse-phase momentum equation by averaging the particle equation of motion directly..." - page 185.

See comments below on ensemble averaging practices using averaging over the infinite volume? As done in their (2.32) and (2.35), (2.37)

for the disperse phase

$$\langle f_D \rangle (\mathbf{x}, t) = \frac{1}{N!\beta_D} \int_{\infty} d\theta^N P(N;t)\, \chi_D\; f_D(\mathbf{x}, t; N), \quad (2.32), \tag{12}$$

where θ^N is the set of vectors of the configuration of the system; this expression for the ensemble averaging further explaned as

$$\langle f_D \rangle (\mathbf{x}, t) = \frac{1}{\beta_D} \int\limits_{|y-x| \leqslant a} d^3y \int\limits_{\infty} d^3w \beta_D^1 \left\langle f_D^{(1)} \right\rangle_1 P(1;t), \quad (2.35), \tag{13}$$

where d^3w is the velocity defferential;

And here also the ensemble average variable $\overline{g}(\mathbf{x}, t)$ over all the configurations such that one particle centre is at $\mathbf{x}$

$$\overline{g}(\mathbf{x}, t) = \frac{1}{n(x,t)(N-1)!} \int\limits_{\infty} d^3w^{(1)} \int\limits_{\infty} d\theta^{N-1} P(N;t) \; g^{(1)}(N;t), \quad (2.37), \tag{14}$$

where $P(N;t)$ is the probability of a specific configuration θ^N of N particles;

$\chi_{C,D}$ is the characteristic function;

β_D^1 is the volume fraction of the disperse phase.

As one neutral observer can suspect from above that the ensemble averaging actually is meaning sence close to the volume averaging !

With one big theoretical disadvantage - infinite integrals should be taken?

There are eventually in the paper very nice coincidences with the VAT averaging theorems results. As the $\nabla\beta_c$ which is the gradient of continuous phase volume fraction in (2.17) (meaning $\langle m \rangle = \beta_c$)

$$\nabla\beta_c = \int\limits_{|x-y|=a} dS_y \mathbf{n} \int\limits_{\infty} d^3w P(1;t) = \int\limits_{|x-y|=a} dS_y \mathbf{n} n(y,t), (2.17), \tag{15}$$

where $n(y,t)$ is local particle number density and $\mathbf{n}$ is the unit normal vector oriented outward from the particles. This relationship quitely resemble and equal to the VAT's

$$\nabla < m > = -\frac{1}{\Delta\Omega} \int\limits_{\partial S_w} \vec{ds} = \frac{1}{\Delta\Omega} \int\limits_{\partial S_w} \vec{ds_1}, \tag{16}$$

see Whitaker's (1984,1994) etc.

Another two theorems from the VAT. One is by Zhang and Prosperetti's (1994) (2.23)

$$\nabla(\beta_c \langle f_c \rangle) = \beta_c \langle \nabla f_c \rangle + \int\limits_{|x-y|=a} \mathbf{n} dS_y \int\limits_{\infty} d^3w P(1;t) \langle f_c \rangle_1 (\mathbf{x}, t \mid 1), \quad (2.23), \tag{17}$$

which has the analog in the VAT theory

$$\nabla(\langle m \rangle \widetilde{p}) = \langle m \rangle \{\nabla p\}_f - \frac{1}{\Delta\Omega} \int\limits_{\partial S_w} p\vec{ds} = \langle m \rangle \{\nabla p\}_f + \frac{1}{\Delta\Omega} \int\limits_{\partial S_w} p\vec{ds_1}, \tag{18}$$

while another is (Zhang and Prosperetti's (1994) (2.24))

$$\nabla \langle f_c \rangle = \langle \nabla f_c \rangle + \frac{1}{\beta_c} \int\limits_{|x-y|=a} \mathbf{n} dS_y \int\limits_{\infty} d^3 w P(1;t) \left[\langle \nabla f_c \rangle_1 (\mathbf{x}, t \mid 1) - \langle \nabla f_c \rangle (\mathbf{x}, t) \right], (2.24), \tag{19}$$

which reminds the VAT's

$$\nabla \widetilde{p} = \{\nabla p\}_f - \frac{1}{\langle m \rangle \Delta\Omega} \int\limits_{\partial S_w} \widehat{p} d\vec{s} = \{\nabla p\}_f + \frac{1}{\Delta\Omega_f} \int\limits_{\partial S_w} \widehat{p} d\vec{s}_1, \tag{20}$$

isn't it ??

Equations of motion of continuous phase are taken as for inviscid fluid

$$\begin{aligned} \nabla \cdot \mathbf{u}_c &= 0, \\ \frac{\partial \mathbf{u}_c}{\partial t} + \nabla \cdot (\mathbf{u}_c \mathbf{u}_c) &= -\frac{1}{\rho_c} \nabla p_c + \mathbf{g}, \end{aligned} \tag{21}$$

where $\mathbf{u}_c$ is the continuous phase velocity. These equations averaged gave the equation

$$\rho_c \frac{\partial}{\partial t} \left(\beta_c \langle \mathbf{u}_c \rangle \right) + \rho_c \nabla \cdot \left(\beta_c \langle \mathbf{u}_c \rangle \langle \mathbf{u}_c \rangle \right) + \beta_c \nabla \langle p_c \rangle =$$

$$= \beta_c \mathbf{A}_c (\mathbf{x}, t) + \rho_c \nabla \cdot \left(\beta_c \mathbf{M}_c \right) + \beta_c \rho_c \mathbf{g}, \ (3.4), \tag{22}$$

where

$$\mathbf{A}_c (\mathbf{x}, t) = \nabla \langle p_c \rangle - \langle \nabla p_c \rangle, \ (3.6), \tag{23}$$

$$\mathbf{A}_c (\mathbf{x}, t) = \frac{1}{\beta_c} \int\limits_{|x-y|=a} \mathbf{n} dS_y \int\limits_{\infty} d^3 w P(1;t) \left[\langle p_c \rangle_1 (\mathbf{x}, t \mid 1) - \langle p_c \rangle (\mathbf{x}, t) \right], \ (3.7), \tag{24}$$

$$\mathbf{M}_c = \langle \mathbf{u}_c \rangle \langle \mathbf{u}_c \rangle - \langle \mathbf{u}_c \mathbf{u}_c \rangle = -\left\langle \left(\mathbf{u}_c - \langle \mathbf{u}_c \rangle \right) \left(\mathbf{u}_c - \langle \mathbf{u}_c \rangle \right) \right\rangle, \quad (3.5), \tag{25}$$

where $\mathbf{M}_c$ is the "we have introduced the Reynolds-like (kinematic) stress tensor. We know that.

They do not have the term reflecting the velocity of the interface surface - which reflects the movements and the change of the relative

position of the interface surface. And they do not have the term with the interface velocity-productions $\frac{1}{\Delta\Omega}\int\limits_{\partial S_w} U_{Cj}U_{Ci}\cdot\overrightarrow{ds}$!!

These can be easily compared to the VAT continuous (fluid) phase momentum equation (without viscous terms) keeping in mind that notations in both methods appeared as close as, for example, $\varrho_f = \rho_c$, $\langle m\rangle = \beta_c$, $p = p_c$, $\langle p_c\rangle = \{p_c\}_f$ and $\beta_c\nabla\langle p_c\rangle = \langle m\rangle\nabla\{p_c\}_f = \langle m\rangle\nabla\widetilde{p}$, also $\widehat{p} = p - \{p\}_f = p - \langle p\rangle$, $\widehat{v} = \left(V - \widetilde{V}\right) = (\mathbf{u}_c - \langle\mathbf{u}_c\rangle)$,

$$\varrho_f\left(\frac{\partial\langle m\rangle\widetilde{V}}{\partial t} + \nabla\cdot\left(\langle m\rangle\widetilde{V}\widetilde{V}\right)\right) + \langle m\rangle\nabla\widetilde{p} =$$

$$= \frac{1}{\Delta\Omega}\int\limits_{\partial S_w}\widehat{p}\overrightarrow{ds}_1 + \varrho_f\nabla\cdot\left(-\langle m\rangle\{\widehat{v}\widehat{v}\}_f\right) + \langle m\rangle\varrho_f\overrightarrow{g}. \qquad (26)$$

Finding out what is the difference $\beta_c\mathbf{A}_c(\mathbf{x},t) = \beta_c(\nabla\langle p_c\rangle - \langle\nabla p_c\rangle) = \langle m\rangle\left(\nabla\widetilde{p} - \{\nabla p\}_f\right)$ to be in the VAT where

$$\langle\nabla p\rangle_f = \nabla\langle p\rangle_f + \frac{1}{\Delta\Omega}\int\limits_{\partial S_w} p\overrightarrow{ds}, \qquad (27)$$

or

$$\langle m\rangle\{\nabla p\}_f = \langle m\rangle\nabla\widetilde{p} + \frac{1}{\Delta\Omega}\int\limits_{\partial S_w}\widehat{p}\overrightarrow{ds}, \qquad (28)$$

so, following these expressions the term $\beta_c\mathbf{A}_c(\mathbf{x},t)$ in VAT notations is

$$\langle m\rangle\left(\nabla\widetilde{p} - \{\nabla p\}_f\right) = -\frac{1}{\Delta\Omega}\int\limits_{\partial S_w}\left(p - \{p\}_f\right)\overrightarrow{ds} = \frac{1}{\Delta\Omega}\int\limits_{\partial S_w}\left(p - \{p\}_f\right)\overrightarrow{ds}_1, \qquad (29)$$

which means that surface integral of pressure fluctuation in VAT and by means of ensemble averaging methodology by Zhang and Prosperetti's (1994) coinsiding

$$\beta_c\mathbf{A}_c(\mathbf{x},t) = \frac{1}{\Delta\Omega}\int\limits_{\partial S_w}\widehat{p}\overrightarrow{ds}_1. \qquad (30)$$

These evaluations and comparison of the (22) and (26) gives impression of exact coinsidence of these equations (especially, if one knows how this volume integrals in (24) being calculated after ensemble averaging process is done !)

The VAT now has been developed to the stage which allows easily get more. For example, this equation in case of viscous effects accounting with only linear right hand part of equation appears in VAT as

$$\varrho_f \left(\frac{\partial \langle m \rangle \widetilde{V}}{\partial t} + \nabla \langle m \rangle \widetilde{V}\widetilde{V} + \nabla \left(\langle m \rangle \{\widehat{v}\widehat{v}\}_f \right) \right) =$$

$$= - \langle m \rangle \nabla \widetilde{p} - \frac{1}{\Delta\Omega} \int\limits_{\partial S_w} \widehat{p}\vec{ds} + \mu \nabla \cdot \nabla \left(\langle m \rangle \widetilde{V} \right) +$$

$$+\mu \nabla \cdot \left[\frac{1}{\Delta\Omega} \int\limits_{\partial S_w} V \cdot \vec{ds} \right] + \frac{\mu}{\Delta\Omega} \int\limits_{\partial S_w} \nabla V \cdot \vec{ds} + \langle m \rangle \varrho_f \vec{g}, \qquad (31)$$

having three more terms due to linear only viscosity phenomena. It seems that the ensemble averaging techniques at present time are not still advanced to this level.

The equation of motion of particle "in the present inviscid framework" is

$$m\dot{\mathbf{w}} = - \int\limits_{|x-z|=a} dS_z \; p_c (\mathbf{z}, t; N) \, \mathbf{n} + m\mathbf{g} + \mathbf{F}_c, \quad (3.9), \qquad (32)$$

where $\dot{\mathbf{w}}$ is the accelaration of particle, N is number of particles, p_c is the pressure field in the continuous phase, m is the (constant) particle mass and $\mathbf{F}_c$ is the force due to collisions with other particles, $\mathbf{n}$ is the unit normal vector oriented outward from the particles.

Particle equation motion in this notation just neglects -

1) effect of $\nabla \langle p_c \rangle$ - gradient of pressure in fluid ;

2) F_{ac} - additional force of relative acceleration of fluid around of particle;

3) F_B - Basset's term influensing the nonsteadyness of fluid flow around the particle (hereditary force);

4) Stokes's law influence on particle's velocity;

5) Effective buoyancy force.

Aso it needs to be said that this equation is not applicable at large Re_p (particle Reynolds number);

This equation used in the conservation equation of number of ensemble realizations ((2.40)) which for the $(m\overline{\mathbf{w}})$

$$\frac{\partial \left(nm\overline{\mathbf{w}} \right)}{\partial t} + \nabla \cdot \left(nm\overline{ww} \right) = nm\overline{\dot{\mathbf{w}}}, \quad (3.10), \qquad (33)$$

where $\overline{w}$ is the average velocity field of the particles' centers of mass, to obtain the averaged momentum equation for the particle with centre of mass at $\mathbf{x}$ which includes fluctuations of velocities term $\mathbf{M}_D$ and term $\mathbf{A}_D$ reflecting the pressure drag over the spheres

$$\rho_D \frac{\partial}{\partial t} (n\overline{w}) + \rho_D \nabla \cdot (n\overline{ww}) = -n\nabla \langle p_c \rangle +$$

$$+n\mathbf{A}_D (\mathbf{x}, t) + \rho_D \nabla \cdot (n\mathbf{M}_D) + n\rho_D \mathbf{g} + \frac{n}{v}\overline{\mathbf{F}_c}, \text{ (3.14)}, \tag{34}$$

$$\mathbf{M}_D = \overline{w}\,\overline{w} - \overline{ww} = -\overline{(w - \overline{w})(w - \overline{w})}, \text{ (3.12)}, \tag{35}$$

$$\mathbf{A}_D (\mathbf{x}, t) = \nabla \langle p_c \rangle - \frac{1}{nv} \int\limits_{|x-z|=a} \mathbf{n} dS_z \int\limits_{\infty} d^3 w P (\mathbf{x}, \mathbf{w}; t) \langle p_c \rangle_1 (\mathbf{z}, t \mid 1), \text{ (3.13)}. \tag{36}$$

0.0.9 Prosperetti's Closures for few cases

p. 195 - "equations derived in the preceding sections contain several terms involving integration over spheres with a radius equal to the particle radius a".

They used the dilute limit small particles situation to "close" the momentum equations. After approximate resolution of continuous phase fluctuation tensor $\mathbf{M}_c$ and vector $\mathbf{A}_D(\mathbf{x}, t)$ through the $\mathbf{M}_D$ they recognize that (p. 199) - "Closure of the system requires an expression for the fluctuating particle volume flux tensor $\mathbf{M}_D$ (see e.g. Drew, 1991). This missing information cannot be supplied internally by the theory without a specification of the initial conditions imposed on the particle probability distribution".

They considered also the case of "finite volume fractions for the linear problem" for which the problem equations were formulated as for inviscid and unconvectional media (eqs. (6.1), (6.2)). (**They do not understand that for the constant volume fraction they do not need to make such a gross assumption).**

For this linear case they consider the comparison with work by Sangani et al. (1991) with further assumption of the "locally uniform pressure gradient" $\mathbf{G}(t)$ which is really present mostly an academical only (limited value) interest (p. 201, ref. to Landau and Lifshitz, 1959).

(p. 203) - **it is the disclosure of everything, watch this** :They describe the numerical algorithm for the **linear case problem without the collision force** doing the ensemble averaging as - "**we first calculate volume averages over the fundamental cell and then average these values over the different realizations. It is the result of this combined average that we identify with the ensemble average $\langle f \rangle$ used in the previous sections"** ??

As long as they consider the spatial uniformity - means homogeneous spatial and statistical distributions - that means they are performing simply the volume averaging, nothing less.

They write (p. 203, eq. (8.2)) - "The volume-averaged fluid acceleration $\widetilde{\dot{\mathbf{u}}}_c$ is obtained directly from a knowledge of $\widetilde{\dot{\mathbf{w}}}$ (also volume averaged!) " using equation

$$\rho_c \widetilde{\dot{\mathbf{u}}}_c = \frac{1}{\Delta\Omega_f} \int\limits_{\Delta\Omega_f} (-\nabla p_c)\, d\omega,$$

which right hand part is the intrinsic phase average of the pressure gradient and equal to

$$\rho_c \widetilde{\dot{\mathbf{u}}}_c = -\nabla \langle p_c \rangle - \frac{1}{\Delta\Omega_f} \int\limits_{\partial S_w} p\vec{ds},$$

closed by Zhang and Prosperetti (1994) as

$$\rho_c \widetilde{\dot{\mathbf{u}}}_c = \frac{1}{\beta_c} \mathbf{G}(t) - \frac{\beta_D}{\beta_c} \rho_D \widetilde{\dot{\mathbf{w}}}, \tag{37}$$

which is nothing than the volume averaged (meaning obtained with VAT) inviscid, linear (unconvectional) simplified equation and which is actually identical to the VAT's very simplified equation (with constant volume fraction -or porosity$\langle m \rangle$)

$$\varrho_f \frac{\partial \widetilde{V}}{\partial t} = -\nabla \{p\}_f - \frac{1}{\Delta\Omega_f} \int\limits_{\partial S_w} \widehat{p}\vec{ds} = -\nabla \{p\}_f - \frac{1}{\Delta\Omega_f} \int\limits_{\partial S_w} p\vec{ds}, \tag{38}$$

here intergand $\widehat{p}$ can be equal to p because of $\nabla \langle m \rangle = 0$ in

$$\frac{1}{\Delta\Omega_f} \int\limits_{\partial S_w} (p - \widehat{p})\, \vec{ds} = \frac{1}{\Delta\Omega_f} \int\limits_{\partial S_w} p\vec{ds} + \frac{\widetilde{p}}{\langle m \rangle} (\nabla \langle m \rangle)\,.$$

~~~~~~~~~~~~~~~~~~~~~~

0.0.10 Zhang, D.Z. and Prosperetti, A., (1994), "Ensemble Phase-Averaged Equations for Bubbly Flows", Phys. Fluids, Vol. 6, No. 9, pp. 2956-2970.

This paper differs from the previously commented mostly in the important one feature - the disperse phase (bubbles) size can be various. **Fluid** still taken as **inviscid**.
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0.0.11 1998th - Conclusions to analysis of work by Prosperetti and co-authors

They presented the most correct methodology of ensemble averaging technique- we do not say that it is correct. We leave this question as open. Some of equations compared with VAT equations shown actual coincidence.

Among of still severe assumptions accepted in the studies by Prosperetti and co-authors should be mentioned:

1) N identical particles, despite their study published on bubbly flow phenomena modeling (Zhang and Prosperetti, 1994b) they treat monodisperse array of particles;.

2) fluid flow is potential.

Particle equation motion taken with neglect of:

3) full effect of $\nabla \langle p_c \rangle$ - gradient of pressure in fluid ;

4) F_{ac} - additional force of relative acceleration of fluid around of particle;

5) F_B - Basset's term influensing the nonsteadyness of fluid flow around the particle (hereditary force);

6) Stokes's law influence on particle's velocity;

7) Effective buoyancy force.

8) Also it needs to be said that this equation is not applicable at large Re_p (particle Reynolds number).

Most of the listed in works by Buyevich assumptions should be also applied toward the develoments by Zhang and Prosperetti, as below are listed:

9) the drag forces exerted by the ambient fluid are linear in the relative fluid velocity (which is applicable to only finest particle suspensions);

10) "all surface tension effects are ignored, so that stresses have no discontinuity at the interface";

11) overlook possible contributions to the effective stresses acting in mean suspension flow of fluctuations of spheres;

12) "spheres to be free from imbedded dipole moments, so that there is no dipole interaction of the spheres between themselves and with a corresponding external field";

13) the relevant and interactive fields of particles positions, velocities, accelerations and angular velocities are taken as independent variables, so that the "strong friction" approximation undertaken to simplify the development, which implies that the only "the positions vectors alone, ... are quite sufficient to characterize possible configurations of the particulate ensemble";

The closure methods were developed for dilute cases (still interestingly enough). They do not know how to close in a real "live" physical situations.

They did actually compared to Drew's works but in exceptional dilute case only. Our comparison to these workers technique and results shown and discussed above.

Comparison with work by Wallis (1991a,b) done substantial. Criticizing Wallis (1991a,b) for the "his use of area and volume, as opposed to ensemble, averaging. Although all these averaging techniques coincide for homogeneous systems, care is needed in interpreting spatial averages for dense, non-homogeneous mixtures whereas ensemble averages are always well defined".

As we clearly shown above in analysis of Buyevich's, Lahey and Drew and co-authors, and Prosperetti and co-authors, this statement is over exaggerating the imaginable advantages of ensemble averaging method if comparing the abilities and results of VAT and ensemble averaging approaches. **The real comparison case by case, situation by situation, feature by feature, equation by equation shows real supremacy of the VAT in fullness, possibilities and real connection to the limiting situations and practical needs.**

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0.0.12 +Marchioro, M., Tanksley, M., and Prosperetti, A., "Mixture Pressure and Stress

in Dispersive Two-Phase Flow," *Intern. J. Multiphase Flow*, **25**, pp. 1395-1429, 1999.

These remarks should be considered as continuation of my 98-99 analysis of the works by Prosperetti and co-authors (see, for example, Travkin and Catton, 1999,2001).

p. 1396 - "However, when the viscosity is large enough, the behavior of the drops would be indistinguishable from that of rigid particles and yet, although the average flow would be exactly the same (it won't be the same !!) in the two cases. the concept of "pressure" inside a rigid particle would be devoid of physical meaning."

This is the artificial argument assentially, because: when you changed your physical problem, the governing equations for that problem would change also, replacing the pressure presense inside of the particle by elasticity governing equations as they said themselves, referring to work by Drew and Lahey, 1993. So, no problem with that.

p. 1396 -"Several authors avoid the introduction of disperse-phase pressure and replace it by an "interfacial pressure", related to the mean continuous-phase pressure in the neighborhood of the particles (see, e.g., Anderson and Jackson, 1967; Ishii, 1975; Drew, 1983; Prosperetti and Jones, 1984; Arnold et al., 1989). It will be shown that this concept is a good approximation to the complete solution to the problem that emerges from our study." ??

Let we see how they proceed with the main arguments and mathematical construction ? For the equation of continuous state

$$\rho_C \left[\frac{\partial \mathbf{u}_C}{\partial t} + \nabla \cdot (\mathbf{u}_C \mathbf{u}_C) \right] = \nabla \cdot \boldsymbol{\sigma}_C - \nabla \psi_C$$

where $\boldsymbol{\sigma}_C$ is the stress, and ψ_C is the potential of the body force.

In case of gravitaion we have $\psi_C = -\rho_C \mathbf{g} \cdot \mathbf{x}$.

But it should be written as $\nabla \psi_C = -\rho_C \mathbf{g} \cdot \mathbf{x}$.

They used to apply averaging separately for each part - left and right, and they have the reason for that.

So the right part becames after averaging as

$$\mathbf{I}_C = \beta_C \left\langle \nabla \cdot \boldsymbol{\sigma}_C \right\rangle - \beta_C \nabla \psi_C, \quad (3),$$

where the angle brackets denote the phase-ensemble average, β_C is the volume fraction of the continuous phase, which they claim could be inhomogeneous in space.

Note that they did not average the potential term $\nabla\psi_C$ correctly? Why ? Prosperetti knows that the gradient term should be averaged not by just multipliyng by β_C ?

It should be as

$$\langle\nabla\psi_C\rangle_C = \nabla\langle\psi_C\rangle_C + \frac{1}{\Delta\Omega}\int\limits_{\partial S_w}\psi_C\vec{ds} = \nabla\left(\beta_C\left\{\psi_C\right\}_C\right) + \frac{1}{\Delta\Omega}\int\limits_{\partial S_w}\psi_C\vec{ds},$$

or in their notations

$$\langle\nabla\psi_C\rangle_C = \nabla\left(\beta_C\langle\psi_C\rangle\right) + \frac{1}{\Delta\Omega}\int\limits_{\partial S_w}\psi_C\vec{ds}.$$

So, their $\langle\rangle$ are my $\{\}_C$.

The left part becames

$$\mathbf{I}_C = \rho_C\left[\frac{\partial}{\partial t}\left(\beta_C\langle\mathbf{u}_C\rangle\right) + \nabla\cdot\left(\beta_C\langle\mathbf{u}_C\mathbf{u}_C\rangle\right)\right], \quad (4)$$

Note, they later in equation (126) - page 1417, excluded the changing in the space spacial property of the volume fraction β_C. It is withdrawn from within the averaging operator $\langle\rangle$? Which is incorrect.

This term should looks like

$$\langle\nabla\cdot\left(\beta_C\mathbf{u}_C\mathbf{u}_C\right)\rangle_C =$$

$$= \nabla < \widetilde{\mathbf{u}}_C\widetilde{\mathbf{u}}_C + \widehat{\mathbf{u}}_C\widehat{\mathbf{u}}_C >_C + \frac{1}{\Delta\Omega}\int\limits_{\partial S_w} U_{Cj}U_{Ci}\cdot\vec{ds} =$$

$$= \nabla < \widetilde{\mathbf{u}}_C\widetilde{\mathbf{u}}_C + \widehat{\mathbf{u}}_C\widehat{\mathbf{u}}_C >_C + \frac{1}{\Delta\Omega}\int\limits_{\partial S_w} U_{Cj}U_{Ci}\cdot\vec{ds} = \quad (39)$$

$$= \nabla < \widetilde{\mathbf{u}}_C\widetilde{\mathbf{u}}_C >_C + \nabla < \widehat{\mathbf{u}}_C\widehat{\mathbf{u}}_C >_C + \frac{1}{\Delta\Omega}\int\limits_{\partial S_w} U_{Cj}U_{Ci}\cdot\vec{ds} = \quad (40)$$

$$= \nabla\left(\beta_C\left\{\widetilde{\mathbf{u}}_C\widetilde{\mathbf{u}}_C\right\}_C\right) + \nabla < \widehat{\mathbf{u}}_C\widehat{\mathbf{u}}_C >_C + \frac{1}{\Delta\Omega}\int\limits_{\partial S_w} U_{Cj}U_{Ci}\cdot\vec{ds} =$$

$$= \nabla\left(\beta_C\left\{\widetilde{\mathbf{u}}_C\widetilde{\mathbf{u}}_C\right\}_C\right) + \nabla\left(\beta_C\left\{\widehat{\mathbf{u}}_C\widehat{\mathbf{u}}_C\right\}_C\right) + \frac{1}{\Delta\Omega}\int\limits_{\partial S_w} U_{Cj}U_{Ci}\cdot\vec{ds},$$

instead their expression means that they do not have the term **with the interface velocity-productions** $\frac{1}{\Delta\Omega}\int\limits_{\partial S_w} U_{Cj}U_{Ci}\cdot\overrightarrow{ds}$!!

And Prosperetii did not have this term in their earlier papers of 94-97.

They explain the averaging of gradient term as (it is unbelivable)

$$\begin{aligned}\beta_C(\mathbf{x})\langle\nabla\cdot\boldsymbol{\sigma}_C(\mathbf{x},t)\rangle &= \nabla\cdot(\beta_C(\mathbf{x})\langle\boldsymbol{\sigma}_C(\mathbf{x},t)\rangle)+\\ &+\int\limits_{|\mathbf{x}-\mathbf{y}|=a} dS_y P(\mathbf{y})\langle\boldsymbol{\sigma}_C(\mathbf{x}|\,\mathbf{y},t)\rangle_1\cdot\mathbf{n}_y,\end{aligned}\tag{41}$$

where $P(\mathbf{y})$ is the single-particle probability density defined in Eq. (A7) and $\langle\boldsymbol{\sigma}_C(\mathbf{x}|\,\mathbf{y},t)\rangle_1$ is the stress at $\mathbf{x}$ averaged conditionally (see the definition (A8)) to the presence of a particle with center at $\mathbf{y}$.

So, first they put the volume fraction function $\beta_C(\mathbf{x})$ (remember it is the space dependable function) out of gradient sign while doing phase averaging of the momentum equation for the continuous phase, then they placed it under the derivative sign ?

Here we need to return, for more preciosion of the description, to their definitions of the averaged variables, functions and operators.

First is the phase average as it was in the paper of 94

$$\langle f_{C,D}\rangle(\mathbf{x},t)=\frac{1}{N!\beta_{C,D}}\int\limits_{\infty} d\theta^N P(N;t)\,\chi_{C,D}(\mathbf{x};N)\;\,f_{C,D}(\mathbf{x},t;N)\,,\quad (A6),\tag{42}$$

with the one particle provability distribution $P(\mathbf{y},\mathbf{w})$ defined as

$$P(1)\equiv P(\mathbf{y},\mathbf{w})=\frac{1}{(N-1)!}\int\limits_{\infty} d\theta^{(N-1)}P(N)\,,\quad (A7)$$

and the one particle conditional average

$$\beta_C^1\langle f_C\rangle_1(\mathbf{x},t|\,\mathbf{y},\mathbf{w})=\frac{1}{(N-1)!}\int\limits_{\infty} d\theta^{(N-1)}\chi_C(\mathbf{x};N)\,f_C(\mathbf{x},t;N)\,P(N-1|\,1)\,,\quad (A8),$$

where the conditional probability $P(N-1|\,1)$ is defined by $P(N)= P(1)\,P(N-1|\,1)$.

Then they used to substitute (page 1398) this averaging expression for the first (and most important) term with the perturbation expansion of the function $\beta_C(\mathbf{x})\langle\nabla\cdot\boldsymbol{\sigma}_C(\mathbf{x},t)\rangle$ into the series

$$\beta_C(\mathbf{x})\left\langle \nabla\cdot\boldsymbol{\sigma}_C(\mathbf{x},t)\right\rangle = \nabla\cdot\left(\beta_C(\mathbf{x})\left\langle\boldsymbol{\sigma}_C(\mathbf{x},t)\right\rangle\right) - -nA\left[\boldsymbol{\sigma}_C\right] + \nabla\cdot\left(\beta_D(\mathbf{x})\,L\left[\boldsymbol{\sigma}_C\right]\right), \quad (6), \tag{43}$$

where n is the particle number density defined in Eq. (A5) and

$$\beta_D(\mathbf{x})\,L\left[\boldsymbol{\sigma}_C\right] = nT\left[\boldsymbol{\sigma}_C\right] + +\nabla\cdot\left\{nI\left[\boldsymbol{\sigma}_C\right] + \nabla\cdot\left[nR\left[\boldsymbol{\sigma}_C\right] + ...\right]\right\}, \quad (7), \tag{44}$$

with

$$A\left[\boldsymbol{\sigma}_C\right](\mathbf{x}) = \overline{\int\limits_{|\mathbf{r}|=a} dS_r\boldsymbol{\sigma}_C\left(\mathbf{x}+\mathbf{r}|\,\mathbf{x},N-1\right)\cdot\mathbf{n}},$$

p.1399 - "Here the overline denotes the particle average defined in Eq. (A9)), i.e., the ensemble average over all the configurations such that one of the particles has center at $\mathbf{x}$; the integration is over the surface of that particle. The terms neglected in Eq. (7) are of higher order in a/L."

$$T\left[\boldsymbol{\sigma}_C\right](\mathbf{x}) = a\overline{\int\limits_{|\mathbf{r}|=a} dS_r\mathbf{n}\left[\boldsymbol{\sigma}_C\left(\mathbf{x}+\mathbf{r}|\,\mathbf{x},N-1\right)\cdot\mathbf{n}\right]},$$

$$I\left[\boldsymbol{\sigma}_C\right](\mathbf{x}) = -\frac{1}{2}a^2\overline{\int\limits_{|\mathbf{r}|=a} dS_r\mathbf{nn}\left[\boldsymbol{\sigma}_C\left(\mathbf{x}+\mathbf{r}|\,\mathbf{x},N-1\right)\cdot\mathbf{n}\right]},$$

$$R\left[\boldsymbol{\sigma}_C\right](\mathbf{x}) = \frac{1}{6}a^3\overline{\int\limits_{|\mathbf{r}|=a} dS_r\mathbf{nnn}\left[\boldsymbol{\sigma}_C\left(\mathbf{x}+\mathbf{r}|\,\mathbf{x},N-1\right)\cdot\mathbf{n}\right]},$$

so we need to assume that equality undertaken

$$\int\limits_{|\mathbf{x}-\mathbf{y}|=a} dS_yP(\mathbf{y})\left\langle\boldsymbol{\sigma}_C(\mathbf{x}|\,\mathbf{y},t)\right\rangle_1\cdot\mathbf{n}_y = nA\left[\boldsymbol{\sigma}_C\right] - \nabla\cdot\left(\beta_D(\mathbf{x})\,L\left[\boldsymbol{\sigma}_C\right]\right) =$$

$$= n\overline{\int\limits_{|\mathbf{r}|=a} dS_r\boldsymbol{\sigma}_C\left(\mathbf{x}+\mathbf{r}|\,\mathbf{x},N-1\right)\cdot\mathbf{n}} -$$

$$-\nabla\cdot\left(na\overline{\int\limits_{|\mathbf{r}|=a} dS_r\mathbf{n}\left[\boldsymbol{\sigma}_C\left(\mathbf{x}+\mathbf{r}|\,\mathbf{x},N-1\right)\cdot\mathbf{n}\right]}\right) +$$

$$+\nabla\nabla\cdot\left(n\frac{1}{2}a^{2}\overline{\int\limits_{|\mathbf{r}|=a}dS_{r}\mathbf{nn}\left[\boldsymbol{\sigma}_{C}\left(\mathbf{x}+\mathbf{r}|\,\mathbf{x},N-1\right)\cdot\mathbf{n}\right]}\right)-$$

$$-\nabla\nabla\nabla\cdot\left(n\frac{1}{6}a^{3}\overline{\int\limits_{|\mathbf{r}|=a}dS_{r}\mathbf{nnn}\left[\boldsymbol{\sigma}_{C}\left(\mathbf{x}+\mathbf{r}|\,\mathbf{x},N-1\right)\cdot\mathbf{n}\right]}\right)-..... \qquad (45)$$

what a terrible expression.
Who would calculate this ?
Then as the logical finished form for this equation it becomes

$$\mathbf{I}_{C}=\nabla\cdot\left(\beta_{C}\left(\mathbf{x}\right)\left\langle\boldsymbol{\sigma}_{C}\left(\mathbf{x},t\right)\right\rangle+\beta_{D}\left(\mathbf{x}\right)L\left[\boldsymbol{\sigma}_{C}\right]\right)-nA\left[\boldsymbol{\sigma}_{C}\right]-\beta_{C}\nabla\psi_{C}, \quad (13).$$

The equation for the disperse phase after the analogous averaging

$$\mathbf{I}_{D}=\beta_{D}\left\langle\nabla\cdot\boldsymbol{\sigma}_{D}\right\rangle-\beta_{D}\nabla\psi_{D}, \quad (15),$$

and then expansion into a Taylor series the gradient of the

$$\begin{aligned}\mathbf{I}_{D} &= nA_{D}\left[\boldsymbol{\sigma}_{D}\right]+\nabla\cdot\overleftrightarrow{\boldsymbol{\Sigma}}_{a}- \\ &\quad -\beta_{D}\nabla\psi_{D}, \quad (16),\end{aligned}$$

where the stress $\overleftrightarrow{\boldsymbol{\Sigma}}_{a}$ is "conceptually similar to L"; its explicit expression is just another series expansion (19) not worth to provide it here. Still that expression includes everywhere the factor $\left\langle\nabla_{r}\cdot\boldsymbol{\sigma}_{D}\right\rangle$, meaning the the average of the stress tensor is still present in the mathematical expressions.

Then authors lamped the both equations together and got the equation they often refer later

$$\mathbf{I}_{C}+\mathbf{I}_{D}=\nabla\cdot\left(\beta_{C}\left\langle\boldsymbol{\sigma}_{C}\right\rangle+\beta_{D}L\left[\boldsymbol{\sigma}_{C}\right]+\overleftrightarrow{\boldsymbol{\Sigma}}_{a}\right)-\beta_{C}\nabla\psi_{C}-\beta_{D}\nabla\psi_{D}, \quad (20).$$

And for all of that in the VAT theory would take the space to denote and calculate in the volume of REV the averaged momentum diffusion term as, for example, is the expression in the non-linear laminar momentum transport equation

$$<\frac{\partial}{\partial x_{j}}\left(2\mu S\right)>_{f}=<\nabla\cdot\left(2\mu S\right)>_{f}=\nabla\cdot\left(<2\mu S>_{f}\right)+$$

$$+\frac{1}{\Delta\Omega}\int\limits_{\partial S_{w}}2\mu S\cdot\vec{ds}=$$

$$\nabla \cdot 2\left[\langle m\rangle \widetilde{\mu}\widetilde{S} + \langle m\rangle \left\{\widehat{\mu}\widehat{S}\right\}_f\right] + \frac{2}{\Delta\Omega}\int\limits_{\partial S_w} \mu S \cdot \vec{ds}, \quad (46)$$

where all those second and third terms are might be presented with the expansions of the very different kinds including one that presented in the paper by Marchioro et al. (1999).

This remark should be kept in mind when considering the usefulness of the ensemble averaging, also the worse remarks will follow.

The result found on the page 1407 - which is the "major result of this paper" - has the following form for the "pressure part" in mixture pressure

$$\begin{aligned} p_m \cong \; & \beta_C(\mathbf{x})\langle p\rangle + \left(1 + \frac{a^2}{10}\nabla^2\right)(n\nu\overline{p}^e) + \\ & + \frac{a^2}{5}\nabla \cdot \left(\overline{n\int\limits_{|\mathbf{r}|=a} dS_r(-p_C)\,\mathbf{n}}\right) + \\ & + \frac{a^2}{14}\nabla\nabla \cdot \left(\overline{n\int\limits_{|\mathbf{r}|=a} dS_r\left(\mathbf{nn} - \frac{1}{3}\mathbf{I}\right)p_C}\right) + ... \end{aligned} \quad (47)$$

where $\overline{p}^e$ is the surface-average of the continuous-phase pressure over the particle surface

$$\overline{p}^e = \frac{1}{4\pi a^2}\int\limits_{|\mathbf{r}|=a} dS_r\,(p_C)\,. \quad (48)$$

Here and everywhere should be used the sign $\cong$ instead of $=$, .because the whole methodology and mathematical expressions are given on the basis of **perturbation expansion of the stress tensor.**

Authors would need still to get rid in the averaged equations of the terms like $\nabla \cdot (\beta_C\langle\boldsymbol{\sigma}_C\rangle)$ and $\nabla \cdot \left(\overleftrightarrow{\boldsymbol{\Sigma}}_a\right)$ which are the averaged stresses.

But they can not do that. They present the most complicated part in the right hand side of momentum equation as

$$\nabla \cdot (\beta_C\langle\boldsymbol{\sigma}_C\rangle + \beta_D L[\boldsymbol{\sigma}_C]) - nA[\boldsymbol{\sigma}_C] = \nabla \cdot \left[-(p_m + q_m)\,\mathbf{I} + \overleftrightarrow{\mathbf{S}} + \overleftrightarrow{\mathbf{A}}_P\right] -$$

$$-\frac{\beta_D}{\nu}A + \frac{a^2}{10}\left[(\nabla n)\times(\nabla\times A) + n\nabla(\nabla\cdot A)\right] + ..., \quad (\mathbf{80}),$$

where the traceless symmetric component $\overleftrightarrow{\mathbf{S}}$ is

$$\overleftrightarrow{\mathbf{S}} = \beta_C\left(\langle\boldsymbol{\sigma}_C\rangle + \langle p_C\rangle \mathbf{I}\right) + \left(1+\frac{a^2}{14}\nabla^2\right)(n\mathbf{t}^s) + \nabla\cdot(n\mathbf{s}^s) + \nabla\nabla\colon\,(n\mathbf{r}^s) -$$
$$-\frac{a^2}{10}n\left[\nabla A + (\nabla A)^T - \frac{2}{3}\mathbf{I}\,(\nabla\cdot A)\right] + ..., \quad (\mathbf{76}),$$

and antisymmetric component $\overleftrightarrow{\mathbf{A}}_P$ is

$$\overleftrightarrow{\mathbf{A}}_{Pji} = \epsilon_{ijk}\frac{1}{2}\left[\overline{n\int\limits_{|\mathbf{r}|=a} dS_r\,(\boldsymbol{\sigma}_C\cdot\mathbf{n})\times\mathbf{r}} - \frac{1}{2}\nabla\cdot\left(\overline{n\int\limits_{|\mathbf{r}|=a} dS_r\mathbf{r}\,((\boldsymbol{\sigma}_C\cdot\mathbf{n})\times\mathbf{r})}\right) + \right.$$
$$\left. + \frac{1}{2}\nabla\nabla\colon\left(\overline{n\int\limits_{|\mathbf{r}|=a} dS_r\mathbf{r}\mathbf{r}\,((\boldsymbol{\sigma}_C\cdot\mathbf{n})\times\mathbf{r})}\right)\right] + ..., \quad (\mathbf{78}),$$

and "the isotropic part of the viscous stress is"

$$q_m = \frac{a^2}{5}\partial_k\left(nA_k^*\right) - \frac{a^2}{14}\partial_k\partial_l\left(nt_{kl}^s\right) + \frac{a^2}{15}n\nabla\cdot A - \partial_k\left(ns_{kmm}^i\right) - \partial_l\partial_k\left(ns_{kmm}^i\right), \quad (79).$$

Page 1417 they gave the final momentum equation for the continuous phase, for example

$$\rho_C\beta_C\frac{\partial}{\partial t}\left(\langle\mathbf{u}_C\rangle\right) + \rho_C\beta_C\langle\mathbf{u}_C\rangle\cdot\nabla\langle\mathbf{u}_C\rangle = -\beta_C\nabla\cdot\left(-p_m\mathbf{I} + \boldsymbol{\Sigma}_C\right)\beta_D\mathbf{f} +$$

$$+\rho_D\nabla\cdot\left(\beta_C\mathbf{M}_C\right) - \beta_C\nabla\psi_C + \frac{a^2}{10}\left[(\nabla n)\times(\nabla\times A) + n\nabla(\nabla\cdot A)\right] + ..., (126), \quad (49)$$

where the kinematic fluctuations induced stress tensor $\mathbf{M}_C$ is given as

$$\mathbf{M}_C = \langle\mathbf{u}_C\rangle\langle\mathbf{u}_C\rangle - \langle\mathbf{u}_C\mathbf{u}_C\rangle, \quad (125),$$

and where the continuous phase viscous contribution to the mixture stress $\boldsymbol{\Sigma}_C$ is

$$\boldsymbol{\Sigma}_C = -q_m\mathbf{I} + \overleftrightarrow{\mathbf{S}} + \overleftrightarrow{\mathbf{A}}_P$$

where q_m is the isotropic part of the viscous stress (which in turn is the another incrediby complicated expression (79)), and $\overleftrightarrow{\mathbf{S}}$ – (76) **and** $\overleftrightarrow{\mathbf{A}}_P$ – (77) are the more incredibly complicated expressions for the symmetric and antisymmetric components of the stress tensor;

and $\mathbf{f}$ is the

$$\mathbf{f} = \frac{1}{\boldsymbol{\nu}} A - \boldsymbol{\nabla}\cdot\left(-p_m\mathbf{I} + \boldsymbol{\Sigma}_C\right), \quad (123).$$

If to substitute these and other expressions into the final momentum equation -

$$\rho_C\beta_C\frac{\partial}{\partial t}\left(\langle\mathbf{u}_C\rangle\right) + \rho_C\beta_C\left\langle\mathbf{u}_C\right\rangle\cdot\nabla\left\langle\mathbf{u}_C\right\rangle = -\beta_C\nabla\cdot\left[-\left(p_m+q_m\right)\mathbf{I}+\right.$$

$$+\beta_C\left(\langle\boldsymbol{\sigma}_C\rangle + \langle p_C\rangle\mathbf{I}\right) + \left(1+\frac{a^2}{14}\nabla^2\right)\left(n\mathbf{t}^s\right) + \nabla\cdot\left(n\mathbf{s}^s\right) + \nabla\nabla:\ \left(n\mathbf{r}^s\right) -$$

$$-\frac{a^2}{10}n\left[\nabla\overline{\int\limits_{|\mathbf{r}|=a} dS_r\boldsymbol{\sigma}_C\left(\mathbf{x}+\mathbf{r}|\,\mathbf{x},N-1\right)\cdot\mathbf{n}} + \right.$$

$$+\left(\nabla\overline{\int\limits_{|\mathbf{r}|=a} dS_r\boldsymbol{\sigma}_C\left(\mathbf{x}+\mathbf{r}|\,\mathbf{x},N-1\right)\cdot\mathbf{n}}\right)^T -$$

$$\left.-\frac{2}{3}\mathbf{I}\left(\nabla\cdot\overline{\int\limits_{|\mathbf{r}|=a} dS_r\boldsymbol{\sigma}_C\left(\mathbf{x}+\mathbf{r}|\,\mathbf{x},N-1\right)\cdot\mathbf{n}}\right)\right] +$$

$$+\epsilon_{ijk}\frac{1}{2}\left[n\overline{\int\limits_{|\mathbf{r}|=a} dS_r\left(\boldsymbol{\sigma}_C\cdot\mathbf{n}\right)\times\mathbf{r}} - \frac{1}{2}\nabla\cdot\left(n\overline{\int\limits_{|\mathbf{r}|=a} dS_r\mathbf{r}\left(\left(\boldsymbol{\sigma}_C\cdot\mathbf{n}\right)\times\mathbf{r}\right)}\right) + \right.$$

$$\left.\left.+\frac{1}{2}\nabla\nabla:\ \left(n\overline{\int\limits_{|\mathbf{r}|=a} dS_r\mathbf{r}\mathbf{r}\left(\left(\boldsymbol{\sigma}_C\cdot\mathbf{n}\right)\times\mathbf{r}\right)}\right)\right]\right] *$$

$$*\left[\beta_D\frac{1}{\boldsymbol{\nu}}\overline{\int\limits_{|\mathbf{r}|=a} dS_r\boldsymbol{\sigma}_C\left(\mathbf{x}+\mathbf{r}|\,\mathbf{x},N-1\right)\cdot\mathbf{n}} - \beta_D\left[\boldsymbol{\nabla}\cdot\left(-\left(p_m+q_m\right)\mathbf{I}+\ +\right.\right.\right.$$

$$+\beta_C\left(\langle\boldsymbol{\sigma}_C\rangle+\langle p_C\rangle\mathbf{I}\right)+\left(1+\frac{a^2}{14}\nabla^2\right)(n\mathbf{t}^s)+\nabla\cdot(n\mathbf{s}^s)+\nabla\nabla:\ (n\mathbf{r}^s)-$$

$$-\frac{a^2}{10}n\left[\nabla\overline{\int\limits_{|\mathbf{r}|=a}dS_r\boldsymbol{\sigma}_C\left(\mathbf{x}+\mathbf{r}|\,\mathbf{x},N-1\right)\cdot\mathbf{n}}+\right.$$

$$+\left(\nabla\overline{\int\limits_{|\mathbf{r}|=a}dS_r\boldsymbol{\sigma}_C\left(\mathbf{x}+\mathbf{r}|\,\mathbf{x},N-1\right)\cdot\mathbf{n}}\right)^T-$$

$$\left.-\frac{2}{3}\mathbf{I}\left(\nabla\cdot\overline{\int\limits_{|\mathbf{r}|=a}dS_r\boldsymbol{\sigma}_C\left(\mathbf{x}+\mathbf{r}|\,\mathbf{x},N-1\right)\cdot\mathbf{n}}\right)\right]+$$

$$+\epsilon_{ijk}\frac{1}{2}\left[n\overline{\int\limits_{|\mathbf{r}|=a}dS_r\left(\boldsymbol{\sigma}_C\cdot\mathbf{n}\right)\times\mathbf{r}}-\frac{1}{2}\nabla\cdot\left(n\overline{\int\limits_{|\mathbf{r}|=a}dS_r\mathbf{r}\left((\boldsymbol{\sigma}_C\cdot\mathbf{n})\times\mathbf{r}\right)}\right)+\right.$$

$$\left.+\frac{1}{2}\nabla\nabla:\ \left(n\overline{\int\limits_{|\mathbf{r}|=a}dS_r\mathbf{r}\mathbf{r}\left((\boldsymbol{\sigma}_C\cdot\mathbf{n})\times\mathbf{r}\right)}\right]\Bigg)\Bigg]\right]+$$

$$+\rho_D\nabla\cdot\left(\beta_C\langle\mathbf{u}_C\rangle\langle\mathbf{u}_C\rangle-\langle\mathbf{u}_C\mathbf{u}_C\rangle\right)-\beta_C\nabla\psi_C+$$

$$+\frac{a^2}{10}\left[(\nabla n)\times\left(\nabla\times\overline{\int\limits_{|\mathbf{r}|=a}dS_r\boldsymbol{\sigma}_C\left(\mathbf{x}+\mathbf{r}|\,\mathbf{x},N-1\right)\cdot\mathbf{n}}\right)+\right.$$

$$\left.+\,n\nabla\left(\nabla\cdot\overline{\int\limits_{|\mathbf{r}|=a}dS_r\boldsymbol{\sigma}_C\left(\mathbf{x}+\mathbf{r}|\,\mathbf{x},N-1\right)\cdot\mathbf{n}}\right)\right]+\ldots.\quad(me).$$

See also comments to this kind of their earlier equation in both papers in (1994) regarding the absence of few important terms - especially important for scacially imhomogeneous and with the second phase also fluid - problems.

As - **They do not have here in this equation - the term reflecting the velocity of the interface surface - which reflects the movements and the change of the relative position of the interface surface. And they do not have the term with the interface velocity-productions** $\frac{1}{\Delta\Omega}\int\limits_{\partial S_w}U_{Cj}U_{Ci}\cdot\overrightarrow{ds}$!!

And definitely on the right hand side they do not have the terms reflecting the processes "on" and "along" of the interface surface. The cause for this is the same - that they can not perform the two times averaging of the right hand side tensorial operators.

But these results (in current paper) and others alike by Prosperetti and co-authors are just the consequences of the perturbation expansion of the stress tensor - because they can not perform the two times averaging in ensemble averaging methodology.

Because the averaging of the right-hand side stress terms in the governing equations requires the averaging the stress tensor and the operator which is the stress tensor itself - meaning the two times averaging.

From this it is of no surprise that the only method of closure they turn to - is the Direct Numerical Simulation of the initial lower level (scale) homogeneous equations.

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0.0.13 +Marchioro, M., Tanksley, M., and Prosperetti, A., "Flow of Spatially Non-Uniform Suspensions.

Part I: Phenomenology," *Intern. J. Multiphase Flow*, **26**, pp. 783-831, 2000.

In this paper - which is just recent - authors continue to apply numerically the developed earlier theory to the Stokes flow.

They again repeat the development of averaged functions and equations - which we studied in their (1999) paper.

Page 800 - ...."**One starts with the particles arranged in a regular array inside the fundamental cell.**"

But the next part will look upon more attentively.

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0.0.14 +Marchioro, M., Tanksley, M., Wang, W., and Prosperetti, A., "Flow of Spatially Non-Uniform Suspensions.

Part II: Systematic Derivation of Closure Relations," *Intern. J. Multiphase Flow*, **27,** pp. 237-276, 2001.

Page 241 - "Our method (of closure) is based on a numerical implementation of the ensemble averaging principle: many realizations of the same macroscopic flow are generated numerically and the results are then averaged."

Goody.

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0.0.15 +Wang, W., and Prosperetti, A., "Flow of Spatially Non-Uniform Suspensions.

Part III: Closure Relations for Porous Media and Spinning Particles," *Intern. J. Multiphase Flow*, **27,** pp. 1627-1653, 2001.

But here they touch the sacred topic - porous media simualtion, let we see.
~~~~~~~~~~~~~~~~~~~~~~~~~~~~~

Page 1633 - "is one in which the particle can not rotate and all ahve the same translational velocity which, without loss of generality, is taken to vanish."

they speak about - "volumetric flow arte " as

$$U^0 = -\frac{2a^2}{9\beta_D^0 \mu_C} K \nabla p_\infty. \quad (4.4)$$

where permeability retained as by Mo and Sangani (1994).

Nothing more then a complicated expressions as in the previous papers.

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0.0.16 Prosperetti, A. and Oquz, H.N., "Physalis: A New 0(N) Method for the Numerical

Simulation of Disperse Systems: Potential Flow of Spheres," *J. Comput. Physics,* **167,** pp. 196-216, 2001.

Here they have their numerical method - page 197 - "Enclose each particle by a surface S_Q (Fig. 1) and assume that the problem at hand is linear or can at elast be approximately linearized in the region of space between the surface of the particle and S_Q."

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0.0.17 Marchioro, M. and Prosperetti, A., "Conduction in Non-Uniform Composites," Proc. R. Soc. London A, ***455,*** *pp. 1483-1508, 1999.*

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0.0.18 Molodtsof, Y. and Muzyka, D.W., (1989), "General Probabilistic

Multiphase Flow Equations for Analyzing Gas-Solids Mixtures", *Int. J. Eng. Fluid Mech.*, Vol. 2, No. 1, pp. 1-24.

In this paper claims the development of gas-solids suspension flow probabilistic equations. Under the word "probabilistic" authors meant the method of mathematical equations derivation using -"The basic variable used in the probabilistic treatment is the phase presence probability. It can be defined by considering N identical experiments carried out on a suspension in a given apparatus..... Given the random nature of the suspension, it would be possible to find, at a point M, at a given time t after the start of one of the experiments, any one of the individual phases making up the mixture."

Since there is no space scale assumed or strictly defined (what does it mean - "would be possible to find" ? To touch ?, to overlap ?) that means - at the point M will be found the particle of the phase i (for example) with the size at least 2 times bigger that the particle actual size !! What to do with that fact and definition ??

Their equations are distinguished in these particular areas:

1) "They take into account the fundamentally random nature of gas-solid mixtures;
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2) They allow the separate consideration of any number of individual phases or groups of phases making up the overall mixture, and supply the necessary conditions for regrouping phases;

3) **They fully account for the direct and indirect contact, and surface and body forces, including forces due to particle-particle collisions,** all of which contribute to the overall flow behavior of gas-solids mixtures".

They write:

"Spatial and time averaging have a disadvantage in that, except under special conditions, the so-called mean variables are not independent of the reference space or reference time period".

But that's correct, that's good! There is nothing wrong with it !

"Spatial and time averaging also both ignore the fundamentally random character of suspension flows".

That's just not true.

"Fortier [6] was the first to introduce the notion of phase presence probability in the treatment of gas-solids suspension flows. Having initially written equations in terms of volume averages for fine particles in suspension, he recognized that, **when the average particle diameter becomes non-negligible with respect to the characteristic dimension of the installation in which the suspension is flowing, this approach can no longer be followed; that is, an appropriate averaging interval no longer exist**".

Why not ?? That's not true.

p. 9: "The total force acting on a phase p (excluding the direct particle-particle interactions, which will be treated in the following section) **within the control volume v,..**"

What is this ? Control volume ??

And later everywhere is "control volume" !

At last let we see their final equations.

Continuity general probabilistic equation for incompressible phase p is

$$\frac{\partial \varrho_p \alpha_p}{\partial t} + \nabla \varrho_p \alpha_p V_{pj} = 0, \tag{50}$$

"incompressible" ?

The general probabilistic momentum equation for the fluid phase is

$$\frac{\partial \varrho_f \alpha_f V_{fi}}{\partial t} + \frac{\partial}{\partial x_j} \varrho_f \alpha_f V_{fi} V_{fj} + \frac{\partial}{\partial x_j} \varrho_f \alpha_f \beta_{ij}^f =$$

$$= \frac{\partial}{\partial x_j} \varrho_f \tau_{ij}^f + F_i^{sf} + \varrho_f \alpha_f g_i \ . \tag{51}$$

This equations are nothing more then some constitutive relationships, with many errors discussed above in analysis of others workers.

Come on, this is worse then even initial by !!

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0.0.19 +Zhao, Y.G., Chan, S.H., and Abou-Ellail, M.M.M., (1994),

"A New Averaging Method for Multiphase, Turbulent Diffusion Flame", in *Proc. Tenth Int. Heat Transfer Conference*, Hewitt , ed., Vol. 2, pp. 189-194.

In this paper the third averaging method in turbulent multiphase flows suggested. It is based on the Favre averaged void fraction $\widetilde{\alpha}$

$$\widetilde{\alpha} = \frac{\overline{\rho\alpha}}{\overline{\rho}},$$

and is done though "the new averaging method defines the mean value of a scalar ϕ_k as"

$$\widehat{\Phi}_k = \frac{\overline{\rho_k \alpha_k \Phi_k}}{\overline{\rho_k}\widetilde{\alpha_k}}, \; k = 1, 2, 3, ...N + 1. \tag{52}$$

Among many incorrectness and simplifications one can list just - a) nonlinearities were ignored and terms treated as with linear variables; b) closures were obtained by an "ad hoc" method - when the possible physical phenomena being described using appropriate coefficients.

Besides, as it is known (see, for example, Ristorcelli and Morrison (1996) that in the equations averaged after Favre "both Favre and Reynolds averaged variables naturally appear and, when mean density gradients are important, not accounting for this distinction contributes to poor results". So, with regard to proposed in work by Zhao et al. (1994) the 3rd averaging method for turbulent variables is of no significant advantage - equations are still the subject of correct derivation and proper closure modeling is a question of existence. .

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0.0.20 +Wallis, G.B., (1982), "Review - Theoretical

Models of Gas-Liquid Flows", J. Fluids Engin., Vol. 104, pp. 279-283.

In this paper author starts with the passage:

"The basic theory of two-phase flow already exists in the form of classical fluid mechanics which describes the details of the motion of either phase. Unfortunately, the application of rigorous reasoning from fundamentals, such as the Navier-Stokes equations, is a hopeless task in all but the most academic examples."

Fortunately, it is not anymore the truth. Regarding the last results obtained in the field of closure of Volume Averaging Theory (VAT) (Whitaker, 199 ; Travkin and Kushch, 1998a,b; Travkin et al., 1998) the problem of closure is actually becoming more the technical question then principal obstacle to application and using of heterogeneous media problem modeling in curtain areas. First of all these are: 1) Solid state multiphase systems heat and mass transport and electrodynamics; 2) Porous media one fluid flow, heat - and mass transfer; 3)

Another good excerpt from that paper which at this time is true:

"As long as consistency and mathematical probity are maintained, there is no more fundamental principle to which one can appeal to determine which version is more correct. The only criterion is utility for the particular purpose for which one is performing the analysis".

That's of no truth in this situation - because people are using the homogeneous Gauss-Ostrogradsky theorems. And later make "conclusions" like Lage and Co about the results.

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0.0.21 Ishii, M., (1975), Thermo-Fluid Dynamic Theory of Two-Phase Flow, Eyrolles, Paris.

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0.0.22 -Ishii, M. and Mishima, K., (1984), "Two-fluid model and

hydrodynamic constitutive relations", Nuclear Engineering and Design, Vol. 82, pp. 107-126 .

Ishii and K. Mishima, "Two-fluid model and hydrodynamic constitutive relations", Nuclear Engineering and Design 82, 107 (1984).

The equations used in these works have often been obtained from two-phase transport modeling equations (Ishii3) with heterogeneity of spacial phase distributions neglected in the bulk. Three-dimensional two-fluid flow equations were obtained by Ishii3 using a statistical averaging method. In his development, he essentially neglected nonlinear phenomena and took the flux forms of the diffusive terms to avoid averaging of the second power differential operators. Ishii and Mishima4 averaged a two-fluid momentum equation of the form

where ■k is the local void fraction, ■i is the mean interfacial shear stress, ■kt is the turbulent stress for the kth phase, —-■ is the averaged viscous stress for the kth phase, ■k is the mass generation and Mik is the generalized interfacial drag. Using the area average in the second time averaging procedure, Ishii and Mishima 4 introduced a distribution of parameters to take into consideration the nonlinearity of convective term averaging. This approach cannot strictly take into account the stochastic character of various kinds of spatial phase distributions.

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0.0.23 -Teyssedou, A. Tapucu, and R. Camarero, "Blocked flow

subchannel simulation comparison with single-phase flow data", *J. Fluids Engin.,* 114, 205 (1992).

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0.0.24 -L. Finson and A. S. Clarke, "The effect of surface roughness

character on turbulent reentry heating", AIAA Paper No. 1459, (1980).

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0.0.25 *W.G. Gray, A. Leijnse, R.L. Kolar, C.A. Blain, Mathematical Tools for Changing Spatial Scales in the*

Analysis of Physical Systems, CRC Press, Boca Raton, 1993.
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0.0.26 *M. Kaviany, Principles of Heat Transfer in Porous*

Media, 2nd. edition, Springer, Berlin, 1995.
Generally it is a good book for the first reading - for students.
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0.0.27 *-U. Khan, W. M. Rohsenow, A. A. Sonin, and et al., "A porous body model for*

predicting temperature distribution in wire-wrapped rod assemblies operating in combined forced and free convection", Nuclear Engineering and Design 35, 199 (1975).
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0.0.28 *S. Whitaker, in Fluid Transport in Porous Media, (Computational Mechanics Publications, Southampton, UK, 1997).*
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Chapter 1
RUSSIAN WORKS

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1.0.29 Bakhvalov, N.S. and Panasenko, G.P. (1989), Homogenization: Averaging Processes in Periodoc Media.

Mathematical Problems in the Mechanics of Composite Materials, Kluwer Acad. Publishers, Dordrecht.

This is the topic I need to approach and spend a time to disclose the difference and issues of compatability and relationaship with VAT. 12/17/99 - but I am working over this since ~~1979. Their book and papers by Russian researchers I studied in USSR at those times. I DID SPENT A LOT OF TIME DOING ANALYSIS of their book and other papers in Russian, but did not write at those times in USSR my conclusions.

1.0.30 Let's do it now. 09-14-2002 - I also spent a lot of time doing analysis and

general estimation of the method. It is still not a scaling method -

p. "XXII" - " The purpose of our resoning is to obtain equations whose coefficients are not rapidly oscillating while their solutions are close to those of the original equations (for appropriate boundary conditions). These new equations are called *averaged equations* (not really - if to bring the VAT), and their coefficients are the effective coefficients of a composite material (actually if to say that the solution is the lower scale coordinates function - but not really in terms of the VAT - on the Upper scale problem description). Sometimes averaging yields equations of a type quite different from the original ones; for example, in §3.4 the averaging of a system of differential equations results in a system of integral-differential equations."

The value L is the characteristic size of a composite specimen and ε is the side of the recurrent cell (medium is a periodic one). It is assumed throughout that the dimensions of the periodoc cell are much smaller than the characteristic size of the specimen: $\varepsilon \ll L$. Let also the length λ be the characteristic spatial size of a problem - that the $l = min\,(L, \lambda)$. Now the assumption that $\varepsilon \ll l$ is also hold.

The problem, for example, of the steady-state heat transfer in the 2D periodic heterogeneous "globular inclusions" medium is stated as Poisson equation
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$$\frac{\partial}{\partial x_1}\left(K_\varepsilon\left(x_1,x_2\right)\frac{\partial u\left(x_1,x_2\right)}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(K_\varepsilon\left(x_1,x_2\right)\frac{\partial u\left(x_1,x_2\right)}{\partial x_2}\right)=f(x_1,x_2), \quad (1.1)$$

$$\begin{array}{rcl} u\left(x_1,x_2\right)|_{\partial S_w-} & = & u\left(x_1,x_2\right)|_{\partial S_w+}, \\ K_\varepsilon\left(x_1,x_2\right)\left.\frac{\partial u\left(x_1,x_2\right)}{\partial n}\right|_{\partial S_w-} & = & K_\varepsilon\left(x_1,x_2\right)\left.\frac{\partial u\left(x_1,x_2\right)}{\partial x_1}\right|_{\partial S_w+}, \end{array} \quad (1.2)$$

where the coefficient $K_\varepsilon\left(x_1,x_2\right)$ is the piece-wise conducticity coefficient in the medium equal to each phase coefficient depending on the location of the point of description.

"then the solution of the problem $u(x_1,x_2)$ of the averaged equation is close to the solution $v_0(x_1,x_2)$ of the averaged equation

$$\begin{array}{rcl} \frac{\partial}{\partial x_1}\left(\widehat{K}\frac{\partial v_0(x_1,x_2)}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(\widehat{K}\frac{\partial v_0(x_1,x_2)}{\partial x_2}\right) & = & f(x_1,x_2), \quad (1.3) \\ \widehat{K} & = & const. \quad (1.4) \end{array}$$

Here $\widehat{K}$ is the composite's effective coefficient of conductivity, with an algorithm for the calculation of $\widehat{K}$ being given in §3."

(Note, here the coordinates used are the same as in the microscopic problem - as on the lower scale if to consider VAT terminology).

So, in our comment we can summarize that this efffective coefficient $\widehat{K}$ is equal to one which defined as

$$\begin{array}{rcl} -\mathbf{q}_{hm}\left(\left(x_1,x_2\right)\right) & = & K_\varepsilon\left(x_1,x_2\right)\nabla u\left(x_1,x_2\right)\cong\widehat{K}\nabla v_0(x_1,x_2), \\ \widehat{K} & = & const, \end{array} \quad (1.5)$$

which is not corresponding to the traditional definition

$$-\left\langle\mathbf{q}_{ht}\left(x_1^u,x_2^u\right)\right\rangle=-\sigma_{ij}^*\left(x_1^u,x_2^u\right)\left\langle\nabla\Phi\left(x_1^u,x_2^u\right)\right\rangle, \quad (1.6)$$

with the really averaged functions - of another scale space and system of coordinates.

P. XXVI - "Solution is sought in the form of series in powers of a small parameter with ε with coefficients depending both on the variables x_i (usually

References in this book: IMPORTANT ONLY - from page XXIV

27. Bogolubov, N.N. and Mitropol'skiĭ, Yu.A., "*Asymptotic Methods in Nonlinear Oscillation Theory,*" Moscow, Nauka, 1974.

36. Butuzov, V.F., Vasil'eva, A.B., and Fedoruk, M.V., "*Asymptotic Methods in Theory of Ordinary Differential Equations,*" Itogi Nauki, Mathematical Analysis, 1967, pp. 5-73.

72. Krylov, N.M. and Bogolubov, N.N., "*Introduction into Nonlinear Mechanics*", Kiev, AN UkSSR, 1937.

73. Kuz'mak, G.E., "Asymptotic Solutions of Nonlinear Differential Equations of the 2nd Order with Inhomogeneous Coefficients," PMM, 1959, Vol. 53, No. 3, pp. 515-526.

88. Lomov, C.A., "*Introduction in General Theory of Singular Perturbations,*" Moscow, Nauka, 1981.

100. Mitropol'skiï, Yu.A., "*The Method of Averaging in Nonlinear Mrchanics,*" Kiev, Naukova Dumka, 1971.

101. Moiseev, N.N., "*Nonlinear Mechanicd Asimptotic Methods,*" Moscow, Nauka, 1969.

103. Naifa, A., "*Perturbation Methods,*" Moscow, Mir, 1976.

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1.0.31 *Bakhvalov, N.S., "Averaged Characteristics of Bodies with Periodic Structures,"*

Dokl. AN USSR, **218**, No.5, pp. 1046-1048, 1974.

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1.0.32 *Bakhvalov, N.S., "Averaging of Partial Differential Equations with Fast Oscillating*

Coefficients," Dokl. AN USSR, **221**, No.3, pp. 516-519, 1975.

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1.0.33 *Bakhvalov, N.S., "Averaging of Nonlinear Partial Differential Equations with Fast*

Oscillating Coefficients," Dokl. AN USSR, **225**, No.2, pp. 249-252, 1975.

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1.0.34 *Bakhvalov, N.S., "On the Sound of Speed in Mixtures,"*

Dokl. AN USSR, No. 6, pp. 1345-1348, 1979.

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1.0.35 *Bakhvalov, N.S. and Eglit, M.E., "Processes in Periodic Media Non-Treatable in Termins of Averaged Characteristics,"*

Dokl. AN USSR, **268**, No. 4, pp. 836-840, 1983.

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1.0.36 *Bakhvalov, N.S., Panasenko, G.P., and Shtaras, A.L., "Homogenization Partial Differential Equations,"*

Encyclopedia Mathematica, 34, Springer, 1988.

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1.0.37 Bensoussan, A., Lions, I.-L., and Papanicolaou, G., "Asymptotic Analysis for Periodic

Structures," North-Holland Publ. Comp., Amsterdam, 1978.

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1.0.38 Berdichevsky, V., (1990), "Heat Transfer in Composite Materials with Stochastic Structure

," in *Heat Transfer 1990, Proceedings of the Ninth International Heat Transfer Conference*, Jerusalem, Israel, Vol. 5, pp. 165-169.

His equation of heat transport is

$$\begin{aligned} \frac{\partial}{\partial x^i} a^{ij} \frac{\partial u}{\partial x^j} &= 0, \ in \ V, \\ u(x) &= u_0(x) \ \ on \ \ \partial V \end{aligned} \tag{1.7}$$

which looks like the upper scale problem's statement. He says on page 165 -"**Today almost nothing is known about the distribution function and the equation for distribution function."**

Then he writes the "average temperature" equation as

$$\begin{aligned} \frac{\partial}{\partial x^i} a^{-ij} \frac{\partial v}{\partial x^j} &= 0, \ in \ V, \ ?? \\ v(x) &= u_0(x) \ \ on \ \ \partial V, \end{aligned} \tag{1.8}$$

the later BC is incorect generally. It might be satisfied - but generally it is incorrect.

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1.0.39 Berdichevsky, V., "On Effective Conductivity of Media with Periodic Inclusions,"

Dokl. AN USSR, **247**, No. 6, pp. 1363-1367, 1979.

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1.0.40 Berdichevsky, V., (1983), Variational Principles of Continuum Mechanics, Moscow, Nauka (in Russian).

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*1.0.41 27. Bogolubov, N.N. and Mitropol'skiĭ, Yu.A., "*Asymptotic Methods in Nonlinear Oscillation Theory,*" Moscow, Nauka, 1974.*

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*1.0.42 48. Dul'nev, G.N., "*Transport Coefficients in Inhomogeneous Media. Thermal Physics Properties of a Matter,*"*

Leningrad Institute of Precise Mechanics and Optics, Leningrad, 1979.

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1.0.43 *49. Dul'nev, G.N. and Zarichnyak, Yu.P.,* "Thermal Conductivity of Mixtures and Composite Materials,*"*

Energiya, Leningrad, 1974.

~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~

1.0.44 *72. Krylov, N.M. and Bogolubov, N.N.,* "Introduction into Nonlinear Mechanics*", Kiev, AN UkSSR, 1937.*

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1.0.45 *100. Mitropol'skiĭ, Yu.A.,* "The Method of Averaging in Nonlinear Mechanics," *Kiev, Naukova Dumka, 1971.*

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1.0.46 *101. Moiseev, N.N.,* "Nonlinear Mechanicd Asimptotic Methods,*" Moscow, Nauka, 1969.*

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1.0.47 *103. Naifa, A., "Perturbation Methods," Moscow, Mir, 1976.*

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1.0.48 *Nigmatulin, R.I., Dynamics of Multiphase Media, Moscow, Nauka, 1987, (in Russian).*

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1.0.49 *Shraiber*

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1.0.50 *Kafarov's gang*

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1.0.51 *Vorotyntsev, M.A. and Kornyshev, A.A., Electrostatics of a Medium with the Spatial Dispersion,*

Moscow, Nauka (in Russian), 1993.

Oh, this is the great book - as an evidence and proof that in microelectrodynamics they already many years as studying the non-local description.!!

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1.0.52 *Comments On the Ensemble (and Volume) Averaging Approaches - Techniques and Dispersed Media Governing Equations Development Using this Approaches in Different Applications*

1.0.53 Ensemble averaging techniques disadvantages:

The major problem of interaction of the processes and phenomena going at each separate site with location of separate element of heterogeneous media can not be resolved completely within the brackets of pure statistical approach of ensemble averaging.
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The fundamental problem with the ensemble averaging method lies in its originally genially non-specific consideration of phenomena.

To make ensemble averaging method workable researchers always need to formulate the final problem for the solution in terms of the spacially specific statements: which means in terms of the originally spacial **Volume Averaging Theory (VAT)** !

Examples of this are numerous, see:

1) Buyevich :

2) Torquato and

3) Zang and Prosperetti:

4)

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1.0.54 Current Volume Avaraging Theory Advantages, Possibilities and Results

It worth and relevent to state here that the VAT at present time is developed to the level that all the above mentioned assumed restrictions (in Buyevich's, Drew and Lahey's, Prosperetti's and works enumerated) can be treated in VAT through the proper mathematical procedures.

The VAT can be applied to transport phenomena in heterogeneous media with the following features:

1) multi-scaled media;

2) media with non-linear physical characteristics;

3) polydisperse morphologies;

4) materials with phase anisotropy;

5) media with non-constant or field dependent phase properties;

6) transient problems;

7) presence of imperfect interface surfaces;

8) presence of internal (mostly at the interface) physico-chemical phenomena, etc,

which is at present moment is of no question to be treat correctly with other heterogeneous medium modeling theories. Examples of that were demonstrated above.

The most common way is to thought that these problems, even in most cases much easier, to be treated through the seeking a solution by doing numerical experiments over more or less the exact morphology of interest. This leads to heavy use of large computers to solve large algebraic statements. The treatment and analysis of the results of such a Direct Numerical Modeling (DNM) is both unappealing and difficult.

And what is acheaved through the DNM should be named as a numerical experiment - no less no more, with all the subsequent impossibility to make general conclusions.

Meanwhile, the VAT presents an incredibly powerful tool for dealing with complex heterogeneous media problems having features like those enumerated above.
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The equations resulting from the use of VAT have strange additional terms that are not usually seen. One needs to ask whether or not these new terms are small enough to ignore. In few strict studies authors managed to show that they are not. In fact, they are in some situatins of the same order of magnitude as the terms that are normally kept.