Published by the Hierarchical Scaled Physics and Technologies (HSPT)
Available online at http://travkin-hspt.com/biophysics

Principles, Biological and Mathematical Modeling For Elasticity, Poroelasticity of Soft Biomedia, Polymers with Fluids Mechanics in the Bioporous Two-scale Media

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Abstract:
This paper aim is to fill the gap in the scaled description of Continuum Mechanics elasticity and elasticity in porous biomedia with phenomena taken for the two scale presentation. Both scales are of continuum mechanics media. Whereas the Fluid Mechanics problems in hard solid porous materials found a great reflection in studies and publications in the last ~30 years, the elastic porous media fluid mechanics has only one short part in publication in this area based on HSP-VAT (Hierarchical Scaled Physics and Volume Averaging Theory). Meanwhile, the biomedia of all continuum mechanics scales that are not fluid(s) are the more or less ductile, heterogeneous, porous and elastic media.
This is the first hard copy paper on the fundamentals of poroelasticity and momentum fluid transport in Heterogeneous continuum two-scale porous soft solid state media, biomedia phenomena theory and modeling. After presentation of the short review of works related to theoretical deductions of "scaled" mathematical formulations seeking dependencies between the strains and stresses on the scale of like "continuum" physics phenomena and of the larger "bulk" scale medium communication of properties, we outline the few conceptual formulations for the two-scale connection of fluid flow and elasticity properties for heterogeneous porous materials.
Following the regular, commonly known path for formulation of Homogeneous medium fluid mechanics and elasticity governing equations we derive in the same way the correct momentum and poroelasticity governing equations for the Upper, bulk, scale in a Heterogeneous biomedium.

Key words: heterogeneous media, polyscale, polyscale modeling, elasticity, heterogeneous elasticity, biomedia

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1. INTRODUCTION

1.1. Historical Observation and Heterogeneous Continuum Mechanics Some Principal Provisions

Most of the initial concepts and fundamental two-phase two-scale heterogeneous media elasticity, poroelasticity dependencies and governing equations were first obtained in 1994-96. Some of them were formulated during research on analytical and numerical simulation of two-phase two-scale HSP-VAT problems in the Ukraine.
We won't be analyzing here the theoretical, modeling trends in the solid state mechanics, particularly in the elasticity theory. We restrict ourselves with a few notes, because the

substantial layer of the research regarding studies in heterogeneous solid state mechanics have been published some years ago in [1].
We wrote there: "The unfortunate fact in the contemporary common knowledge continuum mechanics - The Homogeneous Continuum Mechanics (HCM), is that the treatment of the Heterogeneous Media, Matters being performed as that they are being indistinguishable from the Homogeneous media treatment!
In Continuum Mechanics through the last more than 40 years workers do not recognize the other then the accepted in HCM since ~50ths-60ths Homogeneous mathematical modes (not the theories we can name) by Beran, Hashin, Hill, among others and continued later in 70ths -- 2000ths by Mura, Nemat-Nasser and Hori, we name just a few (many can be mentioned for their sizeable input) English writing authors, with the substantial influence as seems throughout the publications of others."
Note that the ground for Heterogeneous formulations and studies does need to use the different mathematics for Heterogeneous media - Heterogeneous, Hierarchical mathematics of discontinued fields.
The Detailed Micro-Modeling - Direct Numerical Modeling (DMM-DNM) method used predominantly now as the most "full and correct" method for the Upper scale heterogeneous porous media models, is mostly incorrect for this purpose. Especially, when the effective properties and the upper scale characteristics sought as the calculated ones based on these Lower Scale model's solution fields or when compared with experiment or sought to be used for experimental basics.
The DMM-DNM is good for the Lower scale, but averaging and other operations for and over the Upper scale functions, operators, etc. done with the Homogeneous GO theorem are mostly wrong.
On the other hand, we would like to state that the Homogeneous subject physics is just the genuine constituent part of the broader description of the Heterogeneous matter.
Some parts from the above mentioned analytical materials, texts on solid state mechanics, fluid mechanics, elasticity theory for heterogeneous media can be seen with the references in the website - http://www.travkin-hspt.com. From those pertained to the solid state physics and solid state elasticity mechanics texts some recently were also published in the hard copy literature in [2-4].
The Two scale HSP-VAT firstly obtained solutions, the exact ones for the most known common textbooks problems, can be seen in - 1) "Classical Problems in Fluid Mechanics" http://www.travkin-hspt.com/fluid/03.htm; 2) "Classical Problems in Thermal Physics" http://www.travkin-hspt.com/thermph/02.htm; 3) "Globular Morphology Two Scale Electrostatic Exact Solutions" http://www.travkin-hspt.com/eldyn/glob1.htm.
Then were obtained after 2002 the two-scale HSP-VAT even the analytical solutions of the following classical problems: 4) "When the 2x2 is not going to be 4 - What to do?" http://www.travkin-hspt.com/eldyn/WhatToDo2.htm; 5) "Two Scale EM Wave Propagation in Superlattices - 1D Photonic Crystals" http://www.travkin-hspt.com/eldyn/photcrys1.htm; 6) "Two Scale Solution for Acoustic Wave Propagation Through the Multilayer Two-Phase Medium" http://www.travkin-hspt.com/acoustics/supercross.htm; 7) "Effective Coefficients in Electrodynamics" http://www.travkin-hspt.com/eldyn/edeffectivecoeff.htm .
All these cornerstone (and other two-scale) problems have not been solved for Upper (second) Heterogeneous scale since the first half of the XX century by other methods (given in textbooks

the Lower Homogeneous scale "solutions" are wrongly attributed to the Upper Heterogeneous scale averaged fields). These solutions leave no chances for calculations or comparison with experiment of the Upper scale characteristics using the basis of Homogeneous GO. This has no sense, invalid for Heterogeneous problems. The experimental procedures used now are based on the homogeneous medium assumptions.

1.2. Some Definitions of Scaling Related to the Subject of Continuum Physics and Modeling of Heterogeneous Media as Scaled Media

In this publication we will be dealing with the problems of multiscale, heterogeneous, nonlocal and nonlinear character in elasticity for solid and soft solid media.
In each technique employed now in the continuum mechanics elasticity science that has been developed through the periods from their conceptions in XX-th century, a number of assumptions and adjustments have been accepted that are so convenient in use that workers consider them as laws. This has been challenged through the last ~40+ years in different physical sciences, but solid state Hierarchical Scaled elasticity was researched and developed in underground because of the usual conservative mode of progress (bringing in the fundamental change in *modi operandi*) in physics, science generally.
Most of these improvements can be referred to the proper, stricter treatment of collective, interactive phenomena while taking heterogeneous matter for study. To this kind of phenomena/changes we can relate almost any action or process more complicated than collision of "mathematical" ball onto the "mathematical" wall, or movement and collisions of two "mathematical" balls, meaning particles, atoms or molecules in MD.
In all other nature prescribed cases the physical matters are of scaled or multiscale character by existence.
There is no substance of physical content in our known universe that is not a heterogeneous one.
The question is at what scale down the matter is still homogeneous? That answer we don't know yet. And taking the scale an Upper or Lower one, then we will have the Heterogeneous matter anyway.
The volume of the earth can be considered as homogeneous at the galaxy mean scale, meanwhile for our human experiences the earth scale object is a huge heterogeneous one.
Another example - the water which we can obviously consider as a homogeneous matter? While it is not, at an atomic and lower scales.
As always, we need taking into account these physical characteristics of a matter description that always hold and are promoted for future description quality improvements, and sometimes for this quality change.
Also, in physics there is no action or process that we can name a local one, unless we want to. Otherwise, we have to look into the point and what it means more strictly. Obviously, many actions or processes can be separated from their less important, at the moment or case, surroundings or/and forces. But that is always more or less an artificial choice. Also we don't know yet - what is or not the collective influence of the Lower Scale forces, because up to now we connect the scales by approximations with the help of appropriate coefficients. All the physical laws in the past have been developed in this way.
In this paper we won't be concerned to the multiscale, heterogeneous, nonlocal and nonlinear properties in solid state elasticity related to scales that are smaller, than what is used as of the continuum mechanics scales group ~$(10^{-7} – 10^{3})$ m.

Even for these ~ 10 orders of decimal magnitude the conventional homogeneous one scale physical theories provide mostly for the approximate or even ad-hoc adjusting mechanisms for the two-scale Bottom-Up scale communication, and that mode is to be re-entered in the current paper from the Bottom-Up and Top-Down interscale transport (communications) point of view. That says the connections of the scale inherited fields are of great significance/importance. We previously studied thoroughly in many sciences (fields) the contemporary homogeneous physics theories for heterogeneous matter and these reviews are referred below.
The strictest definition for the different scale related fields communication - transformation we suggested in 2004 as the **Scaleportation**.
Scaleportation is the means and procedures of the direct and strict "transformation" of data and processes at one scale to the data and processes of the neighboring Upper or Lower Scale. These interscale communications, scale transformations of data are performed mostly not by formulae using the coefficients as this is customary in homogeneous physics, but via using the interscale governing equations for the phenomena.
Scaleportation has being performed over the all our two-scale solved the HSP problems mentioned in this text and in the website - http://www.travkin-hspt.com, as the simulation methods have been based on algorithms of analytical (exact) or numerical methods created for the direct Bottom-Up (BU) or Top-Down (TD) two-scale solutions.
When more than 2 neighboring scales of physical fields are involved, we have introduced the definition of a **Scaleleaping** (or Leapscaling).
At the same time, we compare and describe in some detail the true multiscaling mechanisms stemmed from the heterogeneous analogs of Gauss-Ostrogradsky theorem and scaled exact governing equations and solutions for classical homogeneous physical problems in different physical discipline fields that are under stable development path through the last more than 40 years.
It is known information that all interatomic forces and consequently the upper meso-scale continuum mechanics phenomena can be explained by electromagnetic forces if starting from the lower atomic scale physics.
That means the attractive interactions named as the van der Waal forces (dipole-dipole and London) and hydrogen bonding, as well as Coulomb long range collective forces, in principle can be evaluated (and will be probably in the near future) via the field generating scaled (two scales $[Sc]$) governing equations that are much more depicting and are of much more accurate description. Thus, and much more difficult in simulation than use of any kind of potentials, Lennand-Jones forces, for example.
We won't count in the present report on the aether potential relating actions and forces. That restriction is obvious at the beginning of polyscale language that establishing for the upper continuum mechanics scales in elasticity of solid state via the undeniable examples, problem considered solutions, etc.
It might help with the understanding of our approach to the more strict physically and mathematically description of many subjects of Heterogeneous, Scaled, and Hierarchical nature, made by nature itself from the atomic, molecular scale that the some knowledge of HSP-VAT (Hierarchical Scaled Physics - Volume Averaging Theory) can be of assistance.
To look through, one might browse our previous studies and analytical reports in other areas where the Heterogeneous, multiphase, scaled media and phenomena are in the core of subject matter, while this should help in estimation of the present solid state Heterogeneous Continuum Mechanics field [5-17] other publications in HSP-VAT.

Some parts from the above mentioned analytical materials, texts have been published also in the hard copy literature, see references in the website -http://www.travkin-hspt.com.
So far, as usual, in almost all the contemporary physics fields, but Fluid Mechanics and parts of Thermal physics, the tools and math used for Heterogeneous, Scaled, Hierarchical description are of the 30-50 years old, from the particle physics, statistical mechanics, and quantum mechanics when the spatial scales used are of ~$(10^{-5} – 10^{-15})$ m and less range.
All these tools of the one scale, homogeneous physics and math, just examples, we have found - have been done with the governing equations that were derived with the use of homogeneous Gauss-Ostrogradsky theorem.
While this is incorrect.
It would be appropriate to remind to young readers, that the similar to present (beginning of XX1-st century) "multiscaling" campaign had happened in the 80-90s of the last century. Then professionals in numbers went into public conferences with claims for a near or almost solved problem of so called "structure-properties" relation. That could not happen, because there was no base for structure-properties phenomenology even. Actually it was, because the VAT started in 1967, nevertheless, the true scaling VAT techniques, advanced mathematics, and solved hierarchical problems appeared only in 80-90s. With not much of interest from few professionals funded in the US and engaged at that time with HSP-VAT research.
That is why the "fashion" for "structure-properties" developments on 80-90s (XX-th century) went away with no results, unlike any other fashion.

2. FUNDAMENTALS OF THE THEORY

2.1 Some principal provisions, conceptual definitions, concepts of scaling matter related to the subject of sub-continuum and continuum physics and modeling of Biological media as the scaled media with physical and mathematical rigor

Principles of the Theory.

1) Principle - Recognition of the fact - that at any scale the biological substance (medium, tissue, sample), a cell or sub-cellular organelles are the Polyphase Polyscale Heterogeneous media. At atomic scale, larger scale, or at some of the continuum scales.
2) Principle - Recognition of the fact - that the structure and the specific "phase" content in a volume with biomedium are the fundamental facts controlling the function of the biomedia along with the environmental (boundary) conditions.
3) Principle - Recognition or knowledge that - in this consideration the polyphase, polyscale studying of the subject - the biomedia sample, tissue, or the sub-cellular structure, organelle, for example, is the most accurate and revealing way of obtaining the facts, conclusions. This was and is one of the main methods - the reductionism (with inseparable up-scale the holism approach) to study the nature by physical and biophysical methods so far for many centuries.
4) Principle - Recognition or knowledge of the fact - that for studying the polyphase matters, media the only correct way is to use the discovered in 60s - 80s methods, theorems of Hierarchical physics and mathematics: Hierarchical Scaled Physics - Volume Averaging Theory, those specifically were created and tuned throughout the last 40+ years for polyphase polyscale physical, biological matters. Of many other methods suggested for the Heterogeneous media theorizing, modeling and understanding, up to this time there is no one more fundamental and/or correct.

We won't discuss or comment on numerous schemes suggested for heterogeneous, physical, biological polyphase media in the last ~100 years, as the mixture theory, for example, because they openly use an inadequate or truly simplified physical and mathematical description of the problem. We also published numerous analytical texts in hard copy and the web publications on this issue.
This fact is not objected or disputed by any professional, openly at least, for the last ~30 years. The complexity and educational policies are preventing professionals to acquire and employ the tools of HSP-VAT.
One of the examples is the use of Gauss-Ostrogradsky (GO) theorem for deduction of governing equations for nearly 200 years in physics and other sciences. This theorem is valid only for a Homogeneous matter, that most of biologists even do not know and acknowledge. In hierarchical mathematics there are many alike theorems for the same purposes that have been developed and used during the last 40 something years.
5) Principle of Physical "Holism" (PPH): A living entity (cell, bacterium, tissue, organ, etc.) should be considered as a system of no "open" physical omission. Meaning - THAT NO ONE FUNCTION, PHYSICAL MECHANISM, PHENOMENON MAY BE OMITTED FROM THE SET OF PROCESSES, PHYSICAL PHENOMENA THAT ARE PRESENT AND VITAL FOR FUNCTIONING OF THIS LIVING ENTITY - MAMMALIAN CELL, PROCARIOTIC CELL, PROTOZOA, SUB-CELLULAR STRUCTURE, A TISSUE WITH LIVE SUPPORT EXCHANGE, ORGAN, ORGANISM.
According to this principle, biologists may not omit in their research, theoretical reasoning, modeling for the reasons of convenience, resource availability, any "reasonable" grounds to save money and/or labor, any known and important for functioning of live entity physical mechanism. As - no omission of phenomena occurring with soft solids, solids and fluids, polyfluids, particles in fluid, scaled fluids, elasticity of solid and soft solid biomedia, mass, energy, heat, momentum, electromagnetism describing phenomena, wave mechanics while studying, modeling, experimenting with the live systems.
Simply because all of the physics mechanisms of known character are participating in functions of the live subject and inseparable from the other functions or mechanisms.
The only admissible path in studying is to substitute the omitted phenomena by the simulating part of the same physical character with principally the same effectual action.

Hierarchical Polyphase Polyscale Concepts for Biomedia Modeling.

1) There is no physical phenomena, process, act that at the determined scale is not of the Volumetric character or is not consciously or sub-consciously understood and/or modeled with the Averaged phenomena of the sub-scale. Any point-like experiment or method, theory is about for accepting at first the premise that the point is formed by/after somehow averaging of the forces/movements/reactions etc. of the below scale physics. Mostly it is done implicitly, even sub-consciously, but this premise is present. The Top-Down sequence of physical phenomena is more accurate down to the sub-atomic scale particles, an aether in between the particles and some pertained to them phenomena.
2) Any Heterogeneous media (Ht) biological sample is in reality a two scale, at least, physical entity that have continuous (meso-scale might be and often is not already continuous, but having discontinuous surfaces, dislocations and other imperfect soft solid state features) transport, physical field processes of a physical volumetric (averaged at some point) character.

3) Any mammalian, plant biomedium, tissue sample or cell or bacterial live cell in reality is a polyscale constantly with continuous transport processes physical volumetric (averaged at some point, not a "point") object (entity), that constitutes a challenge for theorizing and modeling its biological functions.
4) It is not correct to study, model and make valuable definitive conclusions about the biomedium function when using only one scale physical and biological phenomenon(a) in a biosample taken into account. This is the situation now in biology and medicine. Contemporary experimental biological, medical investigations lack the ground and fundamental science, while applying the chemical reactions examination. Chemistry is itself of low understanding of integrative biomolecular mechanisms in a biomedium, tissue, cell, etc.
5) The scale models constitute the separate spatial and temporal physical and mathematical models describing the functions of transport on the scale dependent phenomena in a biomedium.
6) Scale dependent physical (biophysical), mathematical models are inherently communicative Up-and-Down the scales. The scale communication of physical properties has been named as a scaleportation of properties and characteristics in HSP-VAT.
7) It is possible to commute the physical models and medium's properties at the neighboring scales of the biomedium with mathematical strictness using methods developed in HSP-VAT.
8) Each scale developed Homogeneous models in a Ht medium based on Homogeneous presentation of this scale physical definitions and the field's governing equations can be connected more or less mathematically strictly to the neighboring next Up-scale or Down-scale models only (most strictly) via using the interscale mechanisms of HSP-VAT.
9) The polyscale depiction of biomedium's functions, transport, models is the only correct approach for theoretical modeling and simulation, for experiments data reduction when the subject of analysis, theory, model, experiment is the polyscale physically object with the nature prescribed polyscaling of function - as the biological cell, cellular volume, tissue sample, etc.
10) Any Homogeneous biophysical substance, biomedium, tissue or cellular sample is in reality also of the two scale, at least, the physical entity that have molecular, atomic, sub-atomic and continuous description of transport, physical field processes of a physical volumetric character. The averaging and scaleportation of molecular scale transport and phenomena Up- to the Continuum (one of the continuum) scale phenomena and Down- to the molecular, atomic scale, should be used with application of correct Hierarchical mathematics of discontinuous "broken" physical fields.
11) At present, in the last approximately 27-29 years all the main physical fields, definitions and homogeneous governing equations have been reformulated and upgraded using methods of HSP-VAT for acting in the heterogeneous subject matters, for Heterogeneous media of solids, solids and fluids, polyfluids, particles in fluid, scaled fluids, elasticity of solid and soft solid media, mass, heat, momentum, electromagnetism describing phenomena, wave mechanics, acoustical fields. That is now making available the polyscaling in the description and modeling the most obvious and important transport (live) phenomena in a biological medium.
12) Advancements in the last 25 years in Hierarchical Scaled Physics - Volume Averaging Theory allowed to surpass inadequate, incomplete one scale homogeneous theories for averaging (mixing), "bulk" representing theories in almost every field of physics and biology. Now we can formulate the true polyscale Dynamics theories and models for biological objects and processes, Heterogeneous and polyscale by nature and fact.
13) None of the concepts written above is on the lists of "advanced" programs formulated in the recent years (and in "new" formulations of disciplines) in academic, university, government

programs. And that is a signal of misleading development in the contemporary conventional Biology.

2.2. The Local, Non-Local, and Scaled Metrics, Physical Fields, and Their Mathematical Formulation

Claim

> *It is known that for every physical discipline serious mathematical formulation and modeling should have been used the theorem by Gauss-Ostrogradsky (GO). This theorem formulates the connection in physical and mathematical sense via integration between the physical features, phenomena on the closing some volume continuous (outlined by us) surface and the phenomena within the embraced volume.*

That means, it was already implicitly included the definitions of the two spaces and the two coordinate systems - of the initial integration of volume and over the surface and of the limiting (going to collapse the surrounded volume) mathematical expression between both, in the development of this field mathematical equations and models at each of the spaces agreed for consideration.

One is the scale that we perform the spatial integration - it theoretically can collapse into the point? But what is the point? What this point is representing?

The point is the subject without dimension in any direction, if in Cartesian coordinates, other 3D systems of spatial coordinates.

The lower scale (actually the same scale, but non-local) physics variables and properties then silently declared as irrelevant, unimportant - hereby we do not use, mention and mean that the point in our mathematical homogeneous formulation is actually the volume used in the GO theorem to obtain the current equation of interest. This approach is justified as soon as the matter assumed as of a homogeneous nature, because there are the mathematical theorems showing that the limit transitions when the size of a spatial domain collapses to an infinitely small one are presenting the properties of this domain in the selected internal point into which it collapses.

Meanwhile, we know that physics give us the very small objects - that constitute the body of any material - all these objects consist of atom and molecules.

That means, this conjecture for deduction of governing fundamental equations in physics via using the GO theorem where the volume should collapse to the volumeless point is defective.

And this precludes acceptance of the GO theorem for really applying to any physical matter, materials consisting of the atoms and molecules.

Because the GO theorem is mathematically valid for only Homogeneous matter - no atomic and molecular spatial Heterogeneity is accepted.

This surprised "discovery" actually "known" for many decades, since XIXth century, when this approximation was starting to be used in Atomic era physics and chemistry at the beginning of XX century.

There were voices of awareness of this approximation, incorrect for any heterogeneous, polyphase media and materials.

Unfortunately, physicists (leading figures) in the past preferred not to notice this mathematical fact. Chemists followed the path.

Because it was convenient and because this "ignorance" gave the only chance to cover with some mathematical strictness of XIXth century the whole body of physical consideration, more or less strict "proof" of mathematical deductions related to Continuum Media (CM) mechanics.

And most of the media have been accepted as continuum ones, even in chemistry - when it is "known" that any chemical substance, and biological substance and material are discontinuous matters due to existence of their atomic structure.

More correctly is to say that in reality the more precise path for mathematical conventional deductions in physics was and is the way to consider the discontinuous function of physical substance of interest, when it is no way to do otherwise, because of the atomic structure of a matter.

And at the same time, when any mathematical operations need to be performed for that physical material - the tool of operation is the Continuum conjecture - this matter, material is accepted as a continuous one.

Following that premises physicists use the Homogeneous matter GO theorem and alike up to now in almost any physical theory. While this is incorrect.

The most influential and important forces acting on all atomic and Upper scales of physical matter, material are the electromagnetic (EM) forces.

We would accept with precautions in this paper the conventional classical set of Maxwell-Heaviside-Lorenz-Lorentz's (MHLL) EM homogeneous matter governing equations and based on them observed results that brought in and were justified in the HCM. We won't discuss here their validity and meaning.

Nevertheless, we have a prove that the electromagnetic phenomena mathematical expression governing equations used and discussed in many research areas of continuum media is an approximation of valid averaged sub-atomic MHLL governing equations - http://www.travkin-hspt.com/eldyn/maxdown/maxdown.htm "What's Wrong with the Pseudo-Averaging Used in Textbooks on Atomic Physics and Electrodynamics for Maxwell-Heaviside-Lorentz Electromagnetism Equations".

It would be appropriate, meanwhile, to mention that at present exist a few points of view regarding the alternative systems of electromagnetic phenomena developed by different researchers.

There are also the non-local homogeneous media properties. Those also use the basis of integration, or averaging. Integration is taken as for the continuous functions. This legitimate base allows us to approach the variety of media, materials, but only the Homogeneous ones.

For Heterogeneous media and materials there are exist the Heterogeneous Whitaker-Slattery-Anderson-Marle (WSAM) type of theorems reminding the origin from the GO theorem. And the reason for the definitions of at least two scale spaces and physics became the pivotal one.

This, in turn brings the problem of definitions for the point-like Lower and the point-like Upper scale physical fields. Consecutively, we need to define the non-local, averaged fields for stated Heterogeneous problems at their corresponding scales.

We should obey to this original set of reasoning that was done two centuries ago in Physics and in Mathematics for the homogeneous media and translate it onto the heterogeneous, scaled media.

The great issue in all of this is the connection, communication between the physics and properties of each of the two spaces.

While for the homogeneous medium this question is seems never surfaced and discuses with application of the atomic scale discontinuous matter fields, for Heterogeneous media we need to specify these definitions with the greater detail.

After all, this is the key issue of scaling heterogeneous physics as it must work. Thus, we start with the definition of a point, a dot used in the mathematical formulation of physical problems.

Definition 1

At the known and assigned previously system of coordinates the point is the object with no dimension in any of the three coordinates, and this object has the descriptive features, which determine the location of that point in the assigned system of coordinates. The point located physical property of the material has this spatial point determined value.

Following the GO theorem we now know that - if at any point with the coordinates inside of the problem's domain is known the functional dependency for a physical field, which in the most of physical sciences right now is the partial differential or just differential equation(s), then we imply that the domain which served for the derivation of this equation via the GO theorem was the domain of the Lower subspace - because in that theorem we start doing an integration over the finite volume and the finite surface(s).

Now - this principle can be applied to the heterogeneous media, which means - that after the averaging provided according to the one of the WSAM theorems - we get the mathematical equation (dependency) of the higher (another) space. With the corresponding spatial dependencies and the topology of the physical spatial fields.

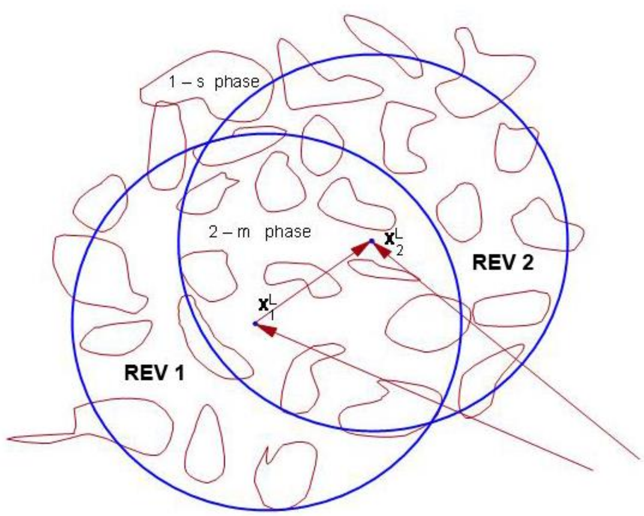

Figure 1. Representative Elementary Volumes (REVs) REV_1 and REV_2 in a heterogeneous medium with the assigned points of representation (x_1, x_2) at the Upper scale physical space. The shape (volumetric form) of the REVs can be of not only spherical one.

With the one substantial different feature - it cannot infinitely be reducing the size, the volume of this domain - because this volume needs to be kept with the most of descriptive features of the both (or more) phases inside of our spatial integration domain.

Definition 2

The Upper Space point when connected to the Lower Scale Domain might reflect, determine, and establish the features of communications between the both space physics. These features can be of different physical description in accordance to their respected space physics definitions. And vise versa - the Higher Space point physics might control, govern, reflect, and determine in some ways the properties of the Lower Scale physical Domain.

It is easier to discuss and argue the features of these definitions right now - after the number of problems in various disciplines of physics were solved in the said mode - when the Lower and Upper scale HSP-VAT governing equations were connected directly before and during their respected solutions - it has been shown by means of the two-scale solutions, especially with the exact two-scale solutions of some common textbooks known classical problems, see in [11] and in the above mentioned other web sections.

2.3. Introductory to Polyscale Description in Physics and Technologies

2.3.1. Hierarchical Scaled Volume Averaging Theory (HSVAT) introductory mathematical definitions, lemmas, and theorems

The basic idea of hierarchical medium description and modeling is to recognize that the physical phenomena, mathematical presentation of those phenomena, and their models can be very different at even neighboring scales. In most of situations those are different even if phenomena themselves are similar or looking as identical, but the scales are different and the lower scale features should be transported to the upper level of description (or Top-Down) - Figure 1. With that action, the useful information from the lower scale physics would be added to the characteristics on the upper scale level.

The following definitions were used in 90s in solid state heterogeneous media elasticity theory (HtET) as well as at the earlier times for other sciences dealing with the scaled heterogeneous problems. The studies by Kushch and co-authors [18-24] will be used here as the example, the only one in the history of HtET as of the consequences of an attempted cooperation with already established mechanic theorist for the studies on HtET, thermal physics and fluid mechanics with implication of the HSP-VAT methods and know-how. It is the useful comparison for establishing the guideline, the path in the Homogeneous elasticity theory that has been used anyway by Kushch throughout these 15-something years studying the subject of HtET with the homogeneous elasticity mechanics methods.

The volume average value of one phase in a two phase composite medium $<s_1(\mathbf{x})>$ in the REV and its fluctuations in various directions, its main physical and mathematical needs, definitions are determined, for example in [3-8,11-13], at first looking simple

$$s_1(\vec{x}) = \left\langle s_1(\vec{x}) \right\rangle + \hat{s}_1(\vec{x}), \qquad \langle s_1 \rangle = \frac{\Delta\Omega_1}{\Delta\Omega}.$$

The three types of two-phase medium averaging over the REV (Figure 1) function f are defined by the following averaging operators arranged in the order of seniority

$$\langle f \rangle = \langle f \rangle_1 + \langle f \rangle_2 = \langle s_1 \rangle \widetilde{f}_1 + (1 - \langle s_1 \rangle)\widetilde{f}_2,$$

where the phase averages are given by

$$\langle f\rangle_1 = \langle s_1\rangle \frac{1}{\Delta\Omega_1} \int\limits_{\Delta\Omega_1} f\left(t,\vec{x}\right) d\omega = \langle s_1\rangle \widetilde{f}_1,$$

$$\langle f\rangle_2 = \langle s_2\rangle \frac{1}{\Delta\Omega_2} \int\limits_{\Delta\Omega_2} f\left(t,\vec{x}\right) d\omega = \langle s_2\rangle \widetilde{f}_2,$$

and the two internal phase averaged functions are given by

$$\{f\}_1 = \widetilde{f}_1 = \frac{1}{\Delta\Omega_1} \int\limits_{\Delta\Omega_1} f\left(t,\vec{x}\right) d\omega,$$

$$\{f\}_2 = \widetilde{f}_2 = \frac{1}{\Delta\Omega_2} \int\limits_{\Delta\Omega_2} f\left(t,\vec{x}\right) d\omega,$$

where $\tilde{f}_1$ is an average over the space of phase one $\Delta\Omega_1$ in the REV, $\tilde{f}_2$ is an average over the second phase volume $\Delta\Omega_2 = \Delta\Omega - \Delta\Omega_1$, and <f> is an average over the whole REV. There are also important averaging theorems for averaging of the spatial ∇ operator - heterogeneous analogs of Gauss-Ostrogradsky theorem. Those are plenty already since 70-80s [25-33]. The first few of them needed to average the field equations are the WSAM theorem (after Whitaker-Slattery-Anderson-Marle) and the one is for the intraphase ∇ averaging. The differentiation theorem for the intraphase averaged function reads

$$\{\nabla f\}_1 = \nabla\widetilde{f} + \frac{1}{\Delta\Omega_1} \int\limits_{\partial S_w} \widehat{f}\ \vec{ds}_1 \ ,$$

$$\widehat{f} = f - \widetilde{f}, \quad f \,\forall\, \Delta\Omega_1,$$

where ∂S_w is the inner surface in the REV, $\vec{ds}_1$ is the second-phase, inward-directed differential area in the REV ($\vec{ds}_1 = \vec{n}_1 dS$).

The WSAM theorem sets the averaged operator ∇ in accordance with

$$\langle\nabla f\rangle_1 = \nabla\langle f\rangle_1 + \frac{1}{\Delta\Omega} \int\limits_{\partial S_{12}} f\ \vec{ds}_1 \ .$$

It can be shown that for the invariable morphology (<m>=const) of the medium the operator $\{\nabla f\}_1$ can be presented also as

$$\{\nabla f\}_1 = \nabla\{f\}_1 + \frac{1}{\Delta\Omega_1} \int\limits_{\partial S_w} f\, \overrightarrow{ds}_1,$$

when <m>=const. Meanwhile, the foundation for averaging made, for example, by Nemat-Nasser and Hori [34] (and many others) is based on conventional homogeneous Gauss-Ostrogradsky theorem (see pp.59-60 in [34]), not of its heterogeneous analogs as the WSAM theorem.

The following averaging theorem has been found for the **rot** operator

$$\langle \nabla \times \mathbf{f} \rangle_1 = \nabla \times \langle \mathbf{f} \rangle_1 + \frac{1}{\Delta\Omega} \int_{\partial S_{12}} d\vec{s}_1 \times \mathbf{f},$$

and as a consequence, the theorem for the intraphase average of ($\nabla\times\mathbf{f}$) is found to be

$$\{\nabla \times \mathbf{f}\}_1 = \nabla \times \{\mathbf{f}\}_1 + \frac{1}{\Delta\Omega_1} \int_{\partial S_{12}} d\vec{s}_1 \times \hat{\mathbf{f}}.$$

The averaged time derivative according to transport theorem forms in the heterogeneous medium the following mathematical equation for a phase one, for example,

$$\left\langle \frac{\partial f}{\partial t} \right\rangle_1 = \frac{\partial}{\partial t}\langle f \rangle_1 - \frac{1}{\Delta\Omega} \int_{\partial S_{12}} (\mathbf{V}_s f) \cdot d\vec{s}_1,$$

where vector $\mathbf{V}_s$ is the velocity of the interface surface ∂S_{12}.

2.3.2. Hierarchical Scaled Volume Averaging Theory (HSVAT) Operating Lemmas

When the interface is fixed in space the averaged functions for the first and second phase (as liquid f and solid s, for example, or two-phase solid) within the REV and over the entire REV fulfill the following conditions, namely

$$\{f + g\}_f = \{f\}_f + \{g\}_f, \qquad \{a\}_f = a,$$

for the conditions of steady state phases

$$\left\{ \frac{\partial f}{\partial t} \right\}_f = \frac{\partial \{f\}_f}{\partial t}, \qquad \left\{ \tilde{f}\, g \right\}_f = \tilde{f}_f\, \tilde{g}_f,$$

where a - a constant, except for the differentiation condition $\{\nabla f\}_1$ and $\langle \nabla f \rangle_1$, that is as written above in the two forms.

There is an important difference in the definitions of averaged and fluctuation values in regards of their meaning and values in the REV comparing to definitions supported by Whitaker and co-authors see, for example, in [32,35]. The treatment and interpretation of the averaged values inside of the REV are supported in the classical interpretation when a value, considered as an averaged inside of the Lower scale REV volume, is still the constant value within the same initial ground scale REV the assigned representation point $\mathbf{x}^u$ for the Upper scale description space. The more detail on that problem are given in [27,31,36]. These methods are supported and verified by the exact two--scale solutions that have been able for performing because of that.

Some clearance to this difficult issue brings the concepts and formulation of the scaled problems in the two or more scales.

The **intrinsic type** of averaging $\{f\}_f$ fulfill all **four** of the above conditions as well as the following **four consequences**

$$\left\{\tilde{f}\right\}_f = \tilde{f}, \quad \left\{\hat{f}\right\}_f = \left\{f - \tilde{f}\right\}_f = 0,$$

$$\left\{\widetilde{f}\,\widetilde{g}\right\}_f = \widetilde{f}_f\,\widetilde{g}_f, \quad \left\{\widetilde{f}\,\widehat{g}\right\}_f = \widetilde{f}_f\,\widetilde{\widehat{g}}_f = 0.$$

At the same time, $\langle f \rangle_f$ and $\langle f \rangle$ do not fulfill neither the second of the averaging conditions for $\{f\}_f$, with equalities

$$\langle f+g\rangle_f = \langle f\rangle_f + \langle g\rangle_f, \qquad \langle a\rangle_f \neq a, \quad \langle a\rangle_f = \langle m\rangle a,$$

while for the stationary morphology spatial volumes

$$\left\langle \frac{\partial f}{\partial t} \right\rangle_f = \frac{\partial \langle f\rangle_f}{\partial t}, \qquad \left\langle \widetilde{f}\, g \right\rangle_f = \widetilde{f}_f \langle g\rangle_f,$$

nor the consequences of the other averaging conditions

$$\left\langle \widetilde{f} \right\rangle_f = \langle m\rangle \widetilde{f}, \Rightarrow \left\langle \widetilde{f} \right\rangle_f \neq \widetilde{f}, \quad \left\langle \widehat{f} \right\rangle_f = \left\langle f - \widetilde{f} \right\rangle_f = 0,$$

$$\left\langle \widetilde{f}\,\widetilde{g} \right\rangle_f = \langle m\rangle \widetilde{f}_f\,\widetilde{g}_f, \Rightarrow \left\langle \widetilde{f}\,\widetilde{g} \right\rangle_f \neq \widetilde{f}_f\,\widetilde{g}_f,$$

$$\left\langle \widetilde{f}\,\widehat{g} \right\rangle_f = \widetilde{f}_f \langle \widehat{g}\rangle_f = 0.$$

At present, the models of transport phenomena in heterogeneous media when using the HSP-VAT allow to treat media with the following features: 1) multi-scaled media; 2) media with non-linear physical characteristics; 3) polydisperse morphologies; 4) materials with phase anisotropy; 5) media with non-constant or field dependent phase properties; 6) transient problems; 7) presence of imperfect interface surfaces; 8) presence of internal (mostly at the interface) physico-chemical phenomena, etc.

More detail on the non-local VAT procedures and governing equations for different physical problems modeled in homogeneous media by linear mathematical physics equations can be found in publications [25-33,37] and many other. Meanwhile, features depicting closure, nonlinear theory, polyphysics applications, polyscale developments, exact solutions, etc. can be found only in the works like [27,31,38-43] and mainly in the website http://www.travkin-hspt.com.

2.4. Averaging, Scale Statements and Scaling Metrics in Homogeneous and Heterogeneous Media Applications

When one would write the recognized in many sciences the convective diffusion equation in a homogeneous medium $\Delta\Omega_f$

$$\frac{\partial C_f}{\partial t} + V \bullet \nabla C_f = D_f \nabla^2 C_f, \tag{1}$$

and the following one obtained in HSP-VAT as for the Upper scale averaged physical properties convective diffusion equation in the same phase $\Delta\Omega_f$ that is the $\Delta\Omega_f = \Delta\Omega - \Delta\Omega_s$

$$\langle m\rangle \frac{\partial \widetilde{C}_f}{\partial t} - \frac{1}{\Delta\Omega} \int_{\partial S_w} (\mathbf{V}_s C_f) \bullet \vec{ds} + \langle m\rangle \widetilde{U}_i \nabla \widetilde{C}_f = -\nabla \left\langle \widehat{C}_f \widehat{U}_i \right\rangle_f + \tag{2}$$

$$+D_f\nabla\bullet\left(\nabla\langle m\rangle\widetilde{C}_f\right)+D_f\nabla\bullet\left[\frac{1}{\Delta\Omega}\int_{\partial S_w} C_f\,\vec{ds}\right]+$$

$$+\frac{D_f}{\Delta\Omega}\int_{\partial S_w}\nabla C_f\bullet\vec{ds},$$

until recent times (80s) it was hardly to be noticed that these equations are perfectly connected and by not only similar notations for concentration and velocity fields, but through the physical scales interaction mainly.

Comparison of Eq. (1) and one of the perfectly legitimate upper scale HSP-VAT heterogeneous medium mass transport Eq. (2) gives a ground for a number of observations which are critical for understanding and proper application of HSP-VAT scaled mathematical models. When one analyzes, for example, Eqs. (1) and (2) it should be up front a clear understanding of what does the global dependence through the interface of coefficient D_f mean ?

1) Is this the piece-wise constant in each of the phase function? Is the jump condition present on an interface?
2) Is this the piece-wise constant function with the relatively smooth but very thin transition layer of interface?
3) The most important question is - Are those interface physics phenomena so important qualitatively or quantitatively that their influence should be found in the governing modeling equations ?

The answers to these questions are laying mostly in the physics of another (smaller) scale and they are not ready for analysis in most of the problems.

Studies of the two-scale problems in HSP-VAT [27,31,38-43,etc.] concerning the thermal macroscale physics and fluid mechanics in capillary and globular heterogeneous (porous) medium morphologies demonstrated that even when interface transport has no openly declared physical special features as, for example, surficial transport - longitudinal diffusion or adsorption, etc., the input of additional surficial and fluctuation terms in the upper scale VAT equations solutions can be significant, reaching the same order in magnitude in balance, as traditional diffusion, heat exchange or friction resistance terms input.

That means - the another non-traditional scaled physical effects are exist and we need to take this into consideration when building the two (or more) scale physical model, experiment, optimize the structure.

More importantly is to have the lower scale physics of the interface included in the upper scale governing equations when this physics is somehow different from physics of upper scale. In electrodynamics those are, for example, effects of polarization on interface surface, surficial longitudinal waves or current, surface plasmons, etc.

Meanwhile, there is no other theory or approach which could include within itself the governing equations formulations of the physical phenomena from neighboring lower or upper scale description. We are not talking here about source terms inclusion. The source term inclusion provided usually in the form of some analytical formulae to describe an effect when there is the

lack of knowledge or resources, or no vital necessity in analyzing the coupled phenomena of different scales.
The most sought after characteristics in heterogeneous media transport which are the effective transport coefficients, can be correctly determined using the conventional definition as
for the steady-state solid two-phase effective diffusivity D^*, for example, with the one of possible formulae

$$-\langle \mathbf{q}_m \rangle = D^* \nabla \langle C \rangle = D_2 \nabla \langle C \rangle + (D_1 - D_2) \frac{1}{\Delta\Omega} \int_{\Delta\Omega_1} \nabla C \, d\omega,$$

but only in the fraction of problems, even while employing the Detailed Micro-Modeling -- Direct Numerical Modeling (DMM-DNM) exact solution that evidently requires the volumetric correct averaging (VAT averaging). The issue is that in a majority of problems, as for inhomogeneous, nonlinear coefficients, for example, and in many transient problems having the two-field, two-phase DMM-DNM exact solution is not enough to find effective coefficients, because the formulation of correct formulae is beyond the homogeneous physics method tools.
Thus, returning to Eqs. (1) and (2) it should be clear understanding that the representative point (RP) of physical field component in Eq. (1) is the dot point of approximate size ~10^3[ηm^3] (for our range of scales), which includes $\geq 0(10^6)$ atoms for a substance with the low molecular mass. Then the reasonable representative elementary HSP-VAT volume (REV) at the upper scale for the Eq. (2) can be about $\gtrsim(1\div10)[\mu m^3]=(10^9\div10^{12})[\eta m^3]$, including within itself the thousands of lower scale RP volumes together with their structural elements, etc.

This REV volume would contain ~$0(10^6)\times(10^9\div10^{12}) \simeq 0(10^{15})$ atoms, which is even for today's upper end high power supercomputers not a solvable MD problem. The important thing is that the algorithms of homogeneous MD techniques are integrating these volumetric effects incorrectly because of using the homogeneous GO theorem [2-3]. That means in a traditional homogeneous MD the Upper scale characteristics won't be analyzed properly anyway.

Still, the main reason, why Eq. (2) together with the lower level micromodel (1) is the more correct tool to model the heterogeneous medium transport, is because it has the mechanisms (additional terms) which clear role is to connect physics of lower scale transport, often different from the upper scale, and morphology of the two-phase (or more) medium to its bulk effective properties and upper scale fields.
It is essential to take into consideration the qualitative transformation of homogeneous conventional formulation of a process under investigation as, for example, the wave propagation problem into heterogeneous scaled (at least on two scales) problem as soon as the numerous structural objects which separate the phases are introduced by nature to the medium for any good reason.

3. ELASTICITY OF SOLID HETEROGENEOUS MEDIA

We can make an averaging of the stress and strain components in the two-phase heterogeneous media to get the "effective" ones by finding the volumetric integrals for the strain $\varepsilon_{ij}(x_k)$ and stress $\sigma_{ij}(x_k)$ tensors

$$\langle \varepsilon_{ij} \rangle = \frac{1}{V} \int_V \varepsilon_{ij}(x_k) dV,$$

$$\langle \sigma_{ij} \rangle = \frac{1}{V} \int_V \sigma_{ij}(x_k) dV,$$

where the REV volume V is written here in a convenient notation used in homogeneous physics.
In the following introductory analysis of HSP-VAT formulations for elasticity theory for the two-scale heterogeneous media description, we would like to use the simple two-phase matrix-particulate phases medium.

3.1 Strain Averaged Over the Two Phases of Heterogeneous Media

At first we would like to start from analysis and comparison of a few averaging operators used for integration within the heterogeneous medium while applying both theorems the Gauss-Ostrogradsky and the WSAM.
We use to compare some results by Kushch et al. [20-24] for heterogeneous medium elasticity when used the GO theorem while the need is to put into operation the WSAM theorem and following two-scale statements with integro-differential equations. Meanwhile, most of analysis now is given in [1] while some notes are published in [3].
In the study by Kushch et al. [22] authors has a pure notion of a feature that can be called as the *scaleblending,* where the hypothesis suggested as such - the remote constant assigned as initial value strain tensor $\langle \varepsilon_{ij} \rangle$ which is in terms of averaged over the both phases, matrix *m* and particular phase *p*, value is

$$\begin{aligned}
\langle \varepsilon_{ij} \rangle &= \langle \varepsilon_{ij} \rangle_m + \langle \varepsilon_{ij} \rangle_p = \langle m \rangle \{\varepsilon_{ij}\}_m + (1-m)\{\varepsilon_{ij}\}_p = \\
&= \langle m \rangle \frac{1}{2\Delta\Omega_m} \int_{\Delta\Omega_m} (u_{i,j} + u_{j,i}) d\omega + (1-m) \frac{1}{2\Delta\Omega_p} \int_{\Delta\Omega_p} (u_{i,j} + u_{j,i}) d\omega = \\
&= \frac{\langle m \rangle}{2} \left(\frac{1}{\Delta\Omega_m} \int_{\Delta\Omega_m} (\nabla \mathbf{u} + (\nabla \mathbf{u})^T) d\omega \right) + \\
&+ \frac{(1-m)}{2} \left(\frac{1}{\Delta\Omega_p} \int_{\Delta\Omega_p} (\nabla \mathbf{u} + (\nabla \mathbf{u})^T) d\omega \right) = \\
&= \frac{\langle m \rangle}{2} \left(\frac{1}{\Delta\Omega_m} \int_{\Delta\Omega_m} \nabla \mathbf{u} d\omega \right) + \frac{\langle m \rangle}{2} \left(\frac{1}{\Delta\Omega_m} \int_{\Delta\Omega_m} (\nabla \mathbf{u})^T d\omega \right) + \\
&+ \frac{(1-m)}{2} \left(\frac{1}{\Delta\Omega_p} \int_{\Delta\Omega_p} (\nabla \mathbf{u}) d\omega \right) + \frac{(1-m)}{2} \left(\frac{1}{\Delta\Omega_p} \int_{\Delta\Omega_p} (\nabla \mathbf{u})^T d\omega \right) =
\end{aligned}$$

when being written in the general form of phase averaged fields has 4 additional surficial terms

$$= \frac{1}{2}\left(\nabla\langle\mathbf{u}\rangle_m + \frac{1}{\Delta\Omega}\int\limits_{\partial S_{wm}} \mathbf{u}\overrightarrow{ds}_m\right) + \frac{1}{2}\left((\nabla\langle\mathbf{u}\rangle_m)^T + \left(\frac{1}{\Delta\Omega}\int\limits_{\partial S_{wm}} \mathbf{u}\overrightarrow{ds}_m\right)^T\right) +$$

$$+\frac{1}{2}\left(\nabla\langle\mathbf{u}\rangle_p + \frac{1}{\Delta\Omega}\int\limits_{\partial S_{wp}} \mathbf{u}\overrightarrow{ds}_p\right) + \frac{1}{2}\left(\left(\nabla\langle\mathbf{u}\rangle_p\right)^T + \left(\frac{1}{\Delta\Omega}\int\limits_{\partial S_{wp}} \mathbf{u}\overrightarrow{ds}_p\right)^T\right),$$

because

$$\frac{1}{\Delta\Omega_m}\int\limits_{\Delta\Omega_m} \nabla\mathbf{u}d\omega = \{\nabla\mathbf{u}\}_m = \frac{1}{\langle m\rangle}\langle\nabla\mathbf{u}\rangle_m = \frac{1}{\langle m\rangle}\left(\nabla\langle\mathbf{u}\rangle_m + \frac{1}{\Delta\Omega}\int\limits_{\partial S_{wm}} \mathbf{u}\overrightarrow{ds}_m\right),$$

and only because and if the boundary conditions of equality of displacements $\mathbf{u}\big|_{\partial S_{wm}} = \mathbf{u}\big|_{\partial S_{wp}}$ and traction vectors $\mathbf{n}_m \cdot \sigma\big|_{\partial S_{wm}} = \mathbf{n}_p \cdot \sigma\big|_{\partial S_{wp}}$ satisfied, which is not valid for many real technological materials, we have the simplification as soon as and only if we consider the strain tensor averaging $\langle \varepsilon_{ij}\rangle = \langle \varepsilon_{ij}\rangle_m + \langle \varepsilon_{ij}\rangle_p$, not the stress averaging $\langle \sigma_{ij}\rangle = \langle \sigma_{ij}\rangle_m + \langle \sigma_{ij}\rangle_p$.

The simplification comes when we can write if displacements at the interface are equal

$$\frac{1}{\Delta\Omega}\int\limits_{\partial S_{wm}} \mathbf{u}\overrightarrow{ds}_m = -\frac{1}{\Delta\Omega}\int\limits_{\partial S_{wp}} \mathbf{u}\overrightarrow{ds}_p, \quad \left(\frac{1}{\Delta\Omega}\int\limits_{\partial S_{wm}} \mathbf{u}\overrightarrow{ds}_m\right)^T = -\left(\frac{1}{\Delta\Omega}\int\limits_{\partial S_{wp}} \mathbf{u}\overrightarrow{ds}_p\right)^T,$$

and we continue to deduct

$$\langle \varepsilon_{ij}\rangle = \langle \varepsilon_{ij}\rangle_m + \langle \varepsilon_{ij}\rangle_p =$$
$$= \frac{1}{2}(\nabla\langle\mathbf{u}\rangle_m) + \frac{1}{2}\left((\nabla\langle\mathbf{u}\rangle_m)^T\right) + \frac{1}{2}\left(\nabla\langle\mathbf{u}\rangle_p\right) + \frac{1}{2}\left(\left(\nabla\langle\mathbf{u}\rangle_p\right)^T\right) =$$
$$= \frac{1}{2}\left(\nabla\langle\mathbf{u}\rangle_m + \nabla\langle\mathbf{u}\rangle_p\right) + \frac{1}{2}\left((\nabla\langle\mathbf{u}\rangle_m)^T + \left(\nabla\langle\mathbf{u}\rangle_p\right)^T\right) =$$
$$= \frac{1}{2}\nabla\left(\langle\mathbf{u}\rangle_m + \langle\mathbf{u}\rangle_p\right) + \frac{1}{2}\nabla\left((\langle\mathbf{u}\rangle_m)^T + \left(\langle\mathbf{u}\rangle_p\right)^T\right) =$$
$$= \frac{1}{2}\nabla(\langle\mathbf{u}\rangle) + \frac{1}{2}\nabla((\langle\mathbf{u}\rangle)^T) = \frac{1}{2}[\nabla(\langle\mathbf{u}\rangle) + \nabla((\langle\mathbf{u}\rangle)^T)] =$$

now, only now we can use the GO theorem because in this expression the averaging is about the whole REV as for homogeneous medium without internal phase divide

$$= \frac{1}{2}\left(\frac{1}{\Delta\Omega}\int\limits_{\partial S_{io}} (\mathbf{nu})_{ij}ds + \frac{1}{\Delta\Omega}\int\limits_{\partial S_{io}} (\mathbf{nu})_{ij}^T ds\right) =$$
$$= \frac{1}{2}\left(\frac{1}{\Delta\Omega}\int\limits_{\partial S_{io}} (n_i u_j + n_j u_i)ds\right),$$

as soon as

$$\overrightarrow{ds}_m = -\overrightarrow{ds}_p, \quad \frac{1}{\Delta\Omega}\int\limits_{\partial S_{wm}} \mathbf{u}\overrightarrow{ds}_m = -\frac{1}{\Delta\Omega}\int\limits_{\partial S_{wp}} \mathbf{u}\overrightarrow{ds}_p,$$

also we need to remember that

$$\nabla\langle\mathbf{u}\rangle_m = \frac{1}{\Delta\Omega}\int\limits_{\partial S_{mio}} \mathbf{u}\overrightarrow{ds}_m, \quad \nabla\langle\mathbf{u}\rangle_p = \frac{1}{\Delta\Omega}\int\limits_{\partial S_{pio}} \mathbf{u}\overrightarrow{ds}_p.$$

It is obtained the form of averaged strain in a full coincidence with the homogeneous elasticity theory because for this mathematical equation for strain there is no difference in expressions for both phases, the averaging is all over the bounding surface, and the interface surface $\partial S_w = \partial S_{wm} = \partial S_{wp}$ physics is of perfect bond.

It can be written also via the surficial integrals of input-output ∂S_{mio} and ∂S_{pio} at the REV's bounding surface $\partial S_{io} = \partial S_{mio} + \partial S_{pio}$,that $\langle \varepsilon_{ij} \rangle$ can be expressed in these conditions of perfect bonds via

$$\begin{aligned}
\langle\varepsilon_{ij}\rangle &= \tfrac{1}{2}\nabla\Big(\big(\langle\mathbf{u}\rangle_m + \langle\mathbf{u}\rangle_p\big) + \big(\nabla\langle\mathbf{u}\rangle_m + \nabla\langle\mathbf{u}\rangle_p\big)^T\Big) = \\
&= \frac{1}{2}\left(\frac{1}{\Delta\Omega}\int\limits_{\partial S_{mio}} \mathbf{u}\overrightarrow{ds}_m\right) + \frac{1}{2}\left(\left(\frac{1}{\Delta\Omega}\int\limits_{\partial S_{mio}} \mathbf{u}\overrightarrow{ds}_m\right)^T\right) + \\
&+ \frac{1}{2}\left(\frac{1}{\Delta\Omega}\int\limits_{\partial S_{pio}} \mathbf{u}\overrightarrow{ds}_p\right) + \frac{1}{2}\left(\left(\frac{1}{\Delta\Omega}\int\limits_{\partial S_{pio}} \mathbf{u}\overrightarrow{ds}_p\right)^T\right) = \\
&= \frac{1}{2}\left(\frac{1}{\Delta\Omega}\int\limits_{\partial S_{mio}} (\mathbf{n}\mathbf{u}_m)_{ij} ds_m\right) + \frac{1}{2}\left(\frac{1}{\Delta\Omega}\int\limits_{\partial S_{mio}} (\mathbf{n}\mathbf{u}_m)^T_{ij} ds_m\right) + \\
&+ \frac{1}{2}\left(\frac{1}{\Delta\Omega}\int\limits_{\partial S_{pio}} (\mathbf{n}\mathbf{u}_p)_{ij} ds_p\right) + \frac{1}{2}\left(\frac{1}{\Delta\Omega}\int\limits_{\partial S_{pio}} (\mathbf{n}\mathbf{u}_p)^T_{ij} ds_p\right) = \\
&= \frac{1}{2}\left(\frac{1}{\Delta\Omega}\int\limits_{\partial S_{io}} (\mathbf{n}\mathbf{u})_{ij} ds + \frac{1}{\Delta\Omega}\int\limits_{\partial S_{io}} (\mathbf{n}\mathbf{u})^T_{ij} ds\right) = \\
&= \frac{1}{2}\left(\frac{1}{\Delta\Omega}\int\limits_{\partial S_{io}} (n_i u_j + n_j u_i) ds\right).
\end{aligned}$$

These mathematical deductions are the pure demonstrations of relationship between the homogeneous medium GO theorem and the WSAM theorem for heterogeneous medium.

That is, when the two (or more) volumetric phases having the interfaces and their volumes are designated to the selected REV, the combined (bulk) mathematical expressions for acting in each volume operator ∇ can be, if summed, equal to the expressions obtained for the whole REV with the GO theorem.

But only for the ∇ operator acting in expressions without any phase coefficient, for selected cases and for perfect interface boundary conditions.

Integration should be done only over the phase intersecting surfaces of bounding (outer) REV surface ∂S_{io} which is

$$\partial S_{io} = \partial S_{mio} + \partial S_{pio}, \quad \partial S_p = \partial S_{wp} + \partial S_{pio}, \quad \partial S_m = \partial S_{wm} + \partial S_{mio},$$
$$\partial S_{wp} = \partial S_{wm} = \partial S_w,$$

but not over all interfaces of the particles in the REV. That is the great difference with the Kushch's et al. [22] mathematics, see also in [1,3].

And finally in the volumetric variables we have the formula for the strain averaged over the two phases

$$\langle \varepsilon_{ij} \rangle = \langle \varepsilon_{ij} \rangle_m + \langle \varepsilon_{ij} \rangle_p = \tfrac{1}{2}\nabla\Big(\big(\langle \mathbf{u} \rangle_m + \langle \mathbf{u} \rangle_p\big) + \big(\nabla\langle \mathbf{u} \rangle_m + \nabla\langle \mathbf{u} \rangle_p\big)^T \Big) =$$
$$= \tfrac{1}{2}\nabla(\langle \mathbf{u} \rangle + (\nabla\langle \mathbf{u} \rangle)^T),$$

while that is only because there are no physical properties assigned to the phases and the perfect interface BC were accepted.

The general form of $\langle \varepsilon_{ij} \rangle$ when used the phase averaged fields and interface surface elasticity properties that are not of perfect traction and displacement conditions is

$$\langle \varepsilon_{ij} \rangle = \langle \varepsilon_{ij} \rangle_m + \langle \varepsilon_{ij} \rangle_p = \langle m \rangle \{\varepsilon_{ij}\}_m + (1-m)\{\varepsilon_{ij}\}_p =$$
$$= \frac{1}{2}\left(\nabla\langle \mathbf{u} \rangle_m + \frac{1}{\Delta\Omega} \int\limits_{\partial S_{wm}} \mathbf{u}\overrightarrow{ds}_m \right) + \frac{1}{2}\left((\nabla\langle \mathbf{u} \rangle_m)^T + \left(\frac{1}{\Delta\Omega} \int\limits_{\partial S_{wm}} \mathbf{u}\overrightarrow{ds}_m \right)^T \right) +$$
$$+ \frac{1}{2}\left(\nabla\langle \mathbf{u} \rangle_p + \frac{1}{\Delta\Omega} \int\limits_{\partial S_{wp}} \mathbf{u}\overrightarrow{ds}_p \right) + \frac{1}{2}\left(\big(\nabla\langle \mathbf{u} \rangle_p\big)^T + \left(\frac{1}{\Delta\Omega} \int\limits_{\partial S_{wp}} \mathbf{u}\overrightarrow{ds}_p \right)^T \right).$$

Note, that we never in need to use this averaging as a separate action from the physics' other aspects of the elasticity problem.

The same can be said and on the averaging in Kushch et al. [20-24] and other works. Nevertheless, when averaged separately over each phase the strain averaged expressions are different.

3.2 Strain Tensor Averaged Over the Matrix Phase

Strain formulation equation over the matrix phase (in a two-phase medium) averaging is

$$\langle \varepsilon_{ij} \rangle_m = \frac{1}{2}\left(\nabla\langle \mathbf{u} \rangle_m + \frac{1}{\Delta\Omega} \int\limits_{\partial S_{wm}} \mathbf{u}\overrightarrow{ds}_m \right) +$$

$$+\frac{1}{2}\left((\nabla\langle\mathbf{u}\rangle_m)^T + \left(\frac{1}{\Delta\Omega} \int\limits_{\partial S_{wm}} \mathbf{u}\overrightarrow{ds}_m \right)^T \right),$$

well, if we recognize that

$$\nabla\langle\mathbf{u}\rangle_m = \frac{1}{\Delta\Omega} \int\limits_{\partial S_{mio}} \mathbf{u}\overrightarrow{ds}_m,$$

then we can write this tensor via the total its interface surface within the REV as the following

$$\langle\varepsilon_{ij}\rangle_m = \frac{1}{2}\left(\frac{1}{\Delta\Omega} \int\limits_{\partial S_{mio}} \mathbf{u}\overrightarrow{ds}_m + \frac{1}{\Delta\Omega} \int\limits_{\partial S_{wm}} \mathbf{u}\overrightarrow{ds}_m \right) +$$

$$+\frac{1}{2}\left(\left(\frac{1}{\Delta\Omega} \int\limits_{\partial S_{mio}} \mathbf{u}\overrightarrow{ds}_m \right)^T + \left(\frac{1}{\Delta\Omega} \int\limits_{\partial S_{wm}} \mathbf{u}\overrightarrow{ds}_m \right)^T \right) =$$

$$= \frac{1}{2}\left(\frac{1}{\Delta\Omega} \int\limits_{\partial S_m} \mathbf{u}\overrightarrow{ds}_m \right) + \frac{1}{2}\left(\frac{1}{\Delta\Omega} \int\limits_{\partial S_m} \mathbf{u}\overrightarrow{ds}_m \right)^T =$$

$$= \frac{1}{2}\left[\frac{1}{\Delta\Omega} \int\limits_{\partial S_m} \left((\mathbf{n}\mathbf{u}_m)_{ij} + (\mathbf{n}\mathbf{u}_m)^T_{ij} \right) ds_m \right] =$$

$$= \frac{1}{2}\left[\frac{1}{\Delta\Omega} \int\limits_{\partial S_m} (n_i u_j + n_j u_i) ds_m \right],$$

via the total interface surface $\partial S_m = \partial S_{wm} + \partial S_{mio}$ for the matrix phase.

3.3 Strain Tensor Averaged Over the Particulate Phase

When strain tensor has been averaged over the another (particulate, or of other morphology) medium it has the formulation

$$\langle\varepsilon_{ij}\rangle_p = \frac{1}{2}\left(\nabla\langle\mathbf{u}\rangle_p + \frac{1}{\Delta\Omega} \int\limits_{\partial S_{wp}} \mathbf{u}\overrightarrow{ds}_p \right) + \frac{1}{2}\left((\nabla\langle\mathbf{u}\rangle_p)^T + \left(\frac{1}{\Delta\Omega} \int\limits_{\partial S_{wp}} \mathbf{u}\overrightarrow{ds}_p \right)^T \right),$$

then we can write this tensor via the total its bounding surface within the REV

$$\langle\varepsilon_{ij}\rangle_p = \frac{1}{2}\left(\frac{1}{\Delta\Omega} \int\limits_{\partial S_{pio}} \mathbf{u}\overrightarrow{ds}_p + \frac{1}{\Delta\Omega} \int\limits_{\partial S_{wp}} \mathbf{u}\overrightarrow{ds}_p \right) +$$

$$+\frac{1}{2}\left(\left(\frac{1}{\Delta\Omega} \int\limits_{\partial S_{pio}} \mathbf{u}\overrightarrow{ds}_p \right)^T + \left(\frac{1}{\Delta\Omega} \int\limits_{\partial S_{wp}} \mathbf{u}\overrightarrow{ds}_p \right)^T \right) =$$

$$= \frac{1}{2}\left[\frac{1}{\Delta\Omega} \int\limits_{\partial S_p} \left((\mathbf{nu}_p)_{ij} + (\mathbf{nu}_p)^T_{ij} \right) ds_p \right] =$$

$$= \frac{1}{2}\left[\frac{1}{\Delta\Omega} \int\limits_{\partial S_p} (n_i u_j + n_j u_i) ds_p \right],$$

where the total bounding surface for the particulate phase $\partial S_p = \partial S_{wp} + \partial S_{pio}$.

Here the interface surface displacement averaged tensor $\langle \mathbf{u}_{ij} \rangle_{\partial S}$ is

$$\langle \mathbf{u}_{ij} \rangle_{\partial S} = \frac{1}{\Delta\Omega} \int\limits_{\partial S_{wm}} \mathbf{u}\overrightarrow{ds}_m = \frac{1}{\Delta\Omega} \int\limits_{\partial S_{wm}} \mathbf{nu} ds_m = \frac{1}{\Delta\Omega} \int\limits_{\partial S_{wm}} (\mathbf{nu}_m)_{ij} ds_m =$$

$$= \frac{1}{\Delta\Omega} \int\limits_{\partial S_{wm}} \left(n_i u_j^{(m)} \right) ds_m.$$

3.4 Stress Tensor Averaging

For the stress relation averaged over the two-phase $\langle \sigma_{ij} \rangle$ we should write the following mathematical expression as soon as

$$\langle \sigma_{ij} \rangle = \langle C^m_{ijkl} \varepsilon^m_{kl} + C^p_{ijkl} \varepsilon^p_{kl} \rangle = \langle C^m_{ijkl} \varepsilon^m_{kl} \rangle_m + \langle C^p_{ijkl} \varepsilon^p_{kl} \rangle_p =$$

$$= \frac{1}{\Delta\Omega} \int\limits_{\Delta\Omega_m} (C^m_{ijkl} \varepsilon^m_{kl}) d\omega + \frac{1}{\Delta\Omega} \int\limits_{\Delta\Omega_p} (C^p_{ijkl} \varepsilon^p_{kl}) d\omega =$$

if C^m_{ijkl} and C^p_{ijkl} are the constant matrixes as in most of the studies accepted, then

$$= \frac{C^m_{ijkl}}{\Delta\Omega} \int\limits_{\Delta\Omega_m} (\varepsilon^m_{kl}) d\omega + C^p_{ijkl} \frac{1}{\Delta\Omega} \int\limits_{\Delta\Omega_p} (\varepsilon^p_{kl}) d\omega =$$

$$= C^m_{ijkl} \langle \varepsilon_{kl} \rangle_m + C^p_{ijkl} \langle \varepsilon_{kl} \rangle_p =$$

$$= \frac{1}{2} C^m_{ijkl} \left[\nabla\langle \mathbf{u} \rangle_m + \frac{1}{\Delta\Omega} \int\limits_{\partial S_{wm}} \mathbf{u}\overrightarrow{ds}_m + (\nabla\langle \mathbf{u} \rangle_m)^T + \left(\frac{1}{\Delta\Omega} \int\limits_{\partial S_{wm}} \mathbf{u}\overrightarrow{ds}_m \right)^T \right] +$$

$$+ \frac{1}{2} C^p_{ijkl} \left[\nabla\langle \mathbf{u} \rangle_p + \frac{1}{\Delta\Omega} \int\limits_{\partial S_{wp}} \mathbf{u}\overrightarrow{ds}_p + \left(\nabla\langle \mathbf{u} \rangle_p\right)^T + \left(\frac{1}{\Delta\Omega} \int\limits_{\partial S_{wp}} \mathbf{u}\overrightarrow{ds}_p \right)^T \right] =$$

$$= \frac{1}{2} C^m_{ijkl} \left[\nabla\langle \mathbf{u} \rangle_m + (\nabla\langle \mathbf{u} \rangle_m)^T + \frac{1}{\Delta\Omega} \int\limits_{\partial S_{wm}} \mathbf{u}\overrightarrow{ds}_m + \left(\frac{1}{\Delta\Omega} \int\limits_{\partial S_{wm}} \mathbf{u}\overrightarrow{ds}_m \right)^T \right] +$$

$$+\frac{1}{2}C^{p}_{ijkl}\left[\nabla\langle\mathbf{u}\rangle_{p}+\left(\nabla\langle\mathbf{u}\rangle_{p}\right)^{T}+\frac{1}{\Delta\Omega}\int\limits_{\partial S_{wp}}\mathbf{u}\overrightarrow{ds}_{p}+\left(\frac{1}{\Delta\Omega}\int\limits_{\partial S_{wp}}\mathbf{u}\overrightarrow{ds}_{p}\right)^{T}\right]=$$

$$=\frac{1}{2}C^{m}_{ijkl}\left[\nabla\langle\mathbf{u}\rangle_{m}+(\nabla\langle\mathbf{u}\rangle_{m})^{T}\right]+\frac{1}{2}C^{m}_{ijkl}\left[\frac{1}{\Delta\Omega}\int\limits_{\partial S_{wm}}\mathbf{u}\overrightarrow{ds}_{m}+\left(\frac{1}{\Delta\Omega}\int\limits_{\partial S_{wm}}\mathbf{u}\overrightarrow{ds}_{m}\right)^{T}\right]+$$

$$+\frac{1}{2}C^{p}_{ijkl}\left[\nabla\langle\mathbf{u}\rangle_{p}+\left(\nabla\langle\mathbf{u}\rangle_{p}\right)^{T}\right]+\frac{1}{2}C^{p}_{ijkl}\left[\frac{1}{\Delta\Omega}\int\limits_{\partial S_{wp}}\mathbf{u}\overrightarrow{ds}_{p}+\left(\frac{1}{\Delta\Omega}\int\limits_{\partial S_{wp}}\mathbf{u}\overrightarrow{ds}_{p}\right)^{T}\right]=$$

$$=\frac{1}{2}C^{m}_{ijkl}\left[\nabla\langle\mathbf{u}\rangle_{m}+(\nabla\langle\mathbf{u}\rangle_{m})^{T}\right]+\frac{1}{2}C^{p}_{ijkl}\left[\nabla\langle\mathbf{u}\rangle_{p}+\left(\nabla\langle\mathbf{u}\rangle_{p}\right)^{T}\right]+$$

plus the 4 surficial terms that appeared as the result of WSAM theorem applied

$$+\frac{1}{2}C^{m}_{ijkl}\left[\frac{1}{\Delta\Omega}\int\limits_{\partial S_{wm}}\mathbf{u}\overrightarrow{ds}_{m}+\left(\frac{1}{\Delta\Omega}\int\limits_{\partial S_{wm}}\mathbf{u}\overrightarrow{ds}_{m}\right)^{T}\right]+$$

$$+\frac{1}{2}C^{p}_{ijkl}\left[\frac{1}{\Delta\Omega}\int\limits_{\partial S_{wp}}\mathbf{u}\overrightarrow{ds}_{p}+\left(\frac{1}{\Delta\Omega}\int\limits_{\partial S_{wp}}\mathbf{u}\overrightarrow{ds}_{p}\right)^{T}\right]=$$

$$=\frac{1}{2}C^{m}_{ijkl}\left[\nabla\langle\mathbf{u}\rangle_{m}+(\nabla\langle\mathbf{u}\rangle_{m})^{T}\right]+\frac{1}{2}C^{p}_{ijkl}\left[\nabla\langle\mathbf{u}\rangle_{p}+\left(\nabla\langle\mathbf{u}\rangle_{p}\right)^{T}\right]+$$

$$+\frac{1}{2}(C^{m}_{ijkl}-C^{p}_{ijkl})\frac{1}{\Delta\Omega}\int\limits_{\partial S_{wm}}\mathbf{u}\overrightarrow{ds}_{m}+\frac{1}{2}(C^{m}_{ijkl}-C^{p}_{ijkl})\left(\frac{1}{\Delta\Omega}\int\limits_{\partial S_{wm}}\mathbf{u}\overrightarrow{ds}_{m}\right)^{T}=$$

$$=\frac{1}{2}C^{m}_{ijkl}\left[\nabla\langle\mathbf{u}\rangle_{m}+(\nabla\langle\mathbf{u}\rangle_{m})^{T}\right]+\frac{1}{2}C^{p}_{ijkl}\left[\nabla\langle\mathbf{u}\rangle_{p}+\left(\nabla\langle\mathbf{u}\rangle_{p}\right)^{T}\right]+$$

$$+\frac{1}{2}(C^{m}_{ijkl}-C^{p}_{ijkl})\left[\frac{1}{\Delta\Omega}\int\limits_{\partial S_{wm}}\mathbf{u}\overrightarrow{ds}_{m}+\left(\frac{1}{\Delta\Omega}\int\limits_{\partial S_{wm}}\mathbf{u}\overrightarrow{ds}_{m}\right)^{T}\right]=\langle\sigma_{ij}\rangle.$$

Thus, we see here 4 more terms (last terms) than would not be expected in homogeneous CM. This is the final equation for the averaged over the two-phase stress tensor $\left\langle\sigma_{ij}\right\rangle$.

The VAT expression for $\left\langle\sigma_{ij}\right\rangle$ generally as for two-phase medium can be written as

$$\langle\sigma_{ij}\rangle=\langle C^{m}_{ijkl}\varepsilon^{m}_{kl}+C^{p}_{ijkl}\varepsilon^{p}_{kl}\rangle=\langle C^{m}_{ijkl}\varepsilon^{m}_{kl}\rangle_{m}+\langle C^{p}_{ijkl}\varepsilon^{p}_{kl}\rangle_{p}=$$

$$=\frac{1}{\Delta\Omega}\int\limits_{\Delta\Omega_{m}}(C^{m}_{ijkl}\varepsilon^{m}_{kl})d\omega+\frac{1}{\Delta\Omega}\int\limits_{\Delta\Omega_{p}}(C^{p}_{ijkl}\varepsilon^{p}_{kl})d\omega=$$

$$= \tfrac{1}{2}C^{m}_{ijkl}\left[\nabla\langle\mathbf{u}\rangle_m + (\nabla\langle\mathbf{u}\rangle_m)^T\right] + \tfrac{1}{2}C^{p}_{ijkl}\left[\nabla\langle\mathbf{u}\rangle_p + \left(\nabla\langle\mathbf{u}\rangle_p\right)^T\right] +$$
$$+\tfrac{1}{2}(C^{m}_{ijkl} - C^{p}_{ijkl})\left[\frac{1}{\Delta\Omega}\int_{\partial S_{wm}}\mathbf{u}\overrightarrow{ds}_m + \left(\frac{1}{\Delta\Omega}\int_{\partial S_{wm}}\mathbf{u}\overrightarrow{ds}_m\right)^T\right].$$

It is relevant to mention here, that because Kushch and co-authors [18-24] mostly avoid writing of the Lower scale elastic equations (EE) and use this assigned "singularity" circumventing artificial REVs - they artificially (for a purpose) derive formulae for $\langle\varepsilon_{kl}\rangle$ and $\langle\sigma_{ij}\rangle$ where there are no surficial terms at all.

The example, we do here for the correct HSP-VAT stress averaging for the elasticity problem written for eucaryotic biocell's Nuclear Pore Complex (NPC) assembly depicted in [44] as for the polyphase medium of particulate nature, where all macromolecular complexes are of the same elastic properties (which is incorrect in general).

When averaged over the whole REV the stress tensor for polyphase medium that all the phases averaging expressions are combined

$$\langle\sigma_{ij}\rangle = \frac{1}{\Delta\Omega}\int_{\Delta\Omega_m}\sigma^{-}_{ij}d\omega + \frac{1}{\Delta\Omega}\sum_{q=1}^{N}\int_{\Delta\Omega_q}\sigma^{+}_{ij}d\omega =$$
$$= \frac{1}{\Delta\Omega}\int_{\Delta\Omega_m}C^{-}_{ijkl}\varepsilon^{-}_{kl}d\omega + \frac{1}{\Delta\Omega}\sum_{q=1}^{N}\int_{\Delta\Omega_q}C^{+}_{ijkl}\varepsilon^{+}_{kl}d\omega -$$
$$-C^{-}_{ijkl}\left(\frac{1}{\Delta\Omega}\right)\sum_{q=1}^{N}\int_{\Delta\Omega_q}\varepsilon^{+}_{kl}d\omega + C^{-}_{ijkl}\left(\frac{1}{\Delta\Omega}\right)\sum_{q=1}^{N}\int_{\Delta\Omega_q}\varepsilon^{+}_{kl}d\omega =$$
$$= C^{-}_{ijkl}\frac{1}{\Delta\Omega}\int_{\Delta\Omega_m}\varepsilon^{-}_{kl}d\omega + \frac{C^{-}_{ijkl}}{\Delta\Omega}\sum_{q=1}^{N}\int_{\Delta\Omega_q}\varepsilon^{+}_{kl}d\omega +$$
$$+(C^{+}_{ijkl} - C^{-}_{ijkl})\frac{1}{\Delta\Omega}\sum_{q=1}^{N}\int_{\Delta\Omega_q}\varepsilon^{+}_{kl}d\omega = C^{-}_{ijkl}\frac{1}{\Delta\Omega}\left(\int_{\Delta\Omega_m}\varepsilon^{-}_{kl}dV + \sum_{q=1}^{N}\int_{\Delta\Omega_q}\varepsilon^{+}_{kl}d\omega\right) +$$
$$+\frac{\left(C^{+}_{ijkl}-C^{-}_{ijkl}\right)}{\Delta\Omega}\sum_{q=1}^{N}\int_{\Delta\Omega_q}\varepsilon^{+}_{kl}d\omega = C^{-}_{ijkl}\langle\varepsilon_{kl}\rangle +$$
$$+\frac{\left(C^{+}_{ijkl}-C^{-}_{ijkl}\right)}{\Delta\Omega}\sum_{q=1}^{N}\int_{\Delta\Omega_q}\varepsilon^{+}_{kl}d\omega =$$
$$= C^{-}_{ijkl}\langle\varepsilon_{kl}\rangle + (C^{+}_{ijkl} - C^{-}_{ijkl})\langle\varepsilon_{kl}\rangle_q,$$

where

$$\langle \varepsilon_{kl} \rangle = \frac{1}{\Delta\Omega}\left(\int\limits_{\Delta\Omega_m} \varepsilon^-_{kl} dV + \sum_{q=1}^{N} \int\limits_{\Delta\Omega_q} \varepsilon^+_{kl} d\omega \right),$$

but only when there is no description of tensor fields is as in the Heterogeneous medium. Now one can continue with the deduction of the polyphase averaged stress Upper scale mathematical model as within the using tools of hierarchical physics description.

At first, would apply the only notations of averaged strains, not of the formulation of these tensors through their own mathematical expressions using displacement vector **u**, one can have

$$\langle \sigma_{ij} \rangle = \frac{1}{\Delta\Omega} \int\limits_{\Delta\Omega_m} \sigma^-_{ij} d\omega + \frac{1}{\Delta\Omega} \sum_{q=1}^{N} \int\limits_{\Delta\Omega_q} \sigma^+_{ij} d\omega =$$

$$= C^-_{ijkl}\left(\langle \varepsilon_{kl} \rangle_m + \sum_{q=1}^{N} \langle \varepsilon_{kl} \rangle_q \right) + (C^+_{ijkl} - C^-_{ijkl}) \sum_{q=1}^{N} \langle \varepsilon_{kl} \rangle_q =$$

$$= \left(C^-_{ijkl}\langle \varepsilon_{kl} \rangle_m + C^-_{ijkl} \sum_{q=1}^{N} \langle \varepsilon_{kl} \rangle_q \right) + (C^+_{ijkl} - C^-_{ijkl}) \sum_{q=1}^{N} \langle \varepsilon_{kl} \rangle_q =$$

$$= C^-_{ijkl}\langle \varepsilon_{kl} \rangle_m + C^+_{ijkl} \sum_{q=1}^{N} \langle \varepsilon_{kl} \rangle_q,$$

where each separate particle or macromolecular assembly (nanoparticle) stress tensor is

$$\langle \varepsilon_{kl} \rangle_q = \frac{1}{\Delta\Omega} \int\limits_{\Delta\Omega_q} \varepsilon^+_{kl} d\omega,$$

here m is the subscript designating solid or soft solid matrix, liquid "matrix" or water solution medium volume all over the REV; N is the number of particles or macromolecular complexes, $\langle \varepsilon_{kl} \rangle_q$ is the strain tensor within the each q-th particle.

This is the usual formula - the Homogeneous Elasticity Theory mathematical expressions (as in Kushch et al. [18-24] and other works) for the two-phase separately homogeneous medium.
To introduce the Lower scale physics of atomic, molecular interactions it needs to agree with the addition of this scale physics phenomena with its physical and mathematical scale models.
Next, when the hierarchical physics (HSP-VAT) applied to the two-phase particulate solid state problem, the formulae for averaged stress tensor $\langle \sigma_{ij} \rangle$ appeared as following

$$\langle \sigma_{ij} \rangle = \frac{1}{2} C^m_{ijkl}\left[\nabla\langle \mathbf{u} \rangle_m + (\nabla\langle \mathbf{u} \rangle_m)^T \right] + \frac{1}{2} \sum_{q=1}^{N} C^q_{ijkl}\left[\nabla\langle \mathbf{u} \rangle_q + \left(\nabla\langle \mathbf{u} \rangle_q \right)^T \right] +$$

one cannot write here $\langle \varepsilon_{kl} \rangle_q$ already, because the averaged $\langle \varepsilon_{kl} \rangle_q$ means and additional surficial terms

$$+\frac{1}{2}C^m_{ijkl}\left[\frac{1}{\Delta\Omega}\int\limits_{\partial S_{wm}}\mathbf{u}\overrightarrow{ds}_m+\left(\frac{1}{\Delta\Omega}\int\limits_{\partial S_{wm}}\mathbf{u}\overrightarrow{ds}_m\right)^T\right]+$$

$$+\frac{1}{2}\sum_{q=1}^{N}C^q_{ijkl}\left[\frac{1}{\Delta\Omega}\int\limits_{\partial S_{wq}}\mathbf{u}\overrightarrow{ds}_q+\left(\frac{1}{\Delta\Omega}\int\limits_{\partial S_{wq}}\mathbf{u}\overrightarrow{ds}_q\right)^T\right],$$

It can be noticed that here there are the *2(N+1)* additional surficial terms - that is the correct averaging for Heterogeneous media.

3.5 Separately in Each Phase the Averaged Stress Tensor $<\sigma_{ij}>$

The following is the initial definition

$$\langle\sigma_{ij}\rangle=\langle C^m_{ijkl}\varepsilon^m_{kl}+C^p_{ijkl}\varepsilon^p_{kl}\rangle=\langle C^m_{ijkl}\varepsilon^m_{kl}\rangle_m+\langle C^p_{ijkl}\varepsilon^p_{kl}\rangle_p=$$

$$=\frac{1}{\Delta\Omega}\int\limits_{\Delta\Omega_m}(C^m_{ijkl}\varepsilon^m_{kl})d\omega+\frac{1}{\Delta\Omega}\int\limits_{\Delta\Omega_p}(C^p_{ijkl}\varepsilon^p_{kl})d\omega=$$

if C^m_{ijkl} and C^p_{ijkl} are constants as in all Kushch' studies, for example, in [18-24], then

$$=\frac{C^m_{ijkl}}{\Delta\Omega}\int\limits_{\Delta\Omega_m}(\varepsilon^m_{kl})d\omega+C^p_{ijkl}\frac{1}{\Delta\Omega}\int\limits_{\Delta\Omega_p}(\varepsilon^p_{kl})d\omega=$$

$$=C^m_{ijkl}\langle\varepsilon_{kl}\rangle_m+C^p_{ijkl}\langle\varepsilon_{kl}\rangle_p.$$

Now in each phase - in matrix $\langle\sigma_{ij}\rangle_m$ is

$$\langle\sigma_{ij}\rangle_m=\langle C^m_{ijkl}\varepsilon^m_{kl}\rangle_m=C^m_{ijkl}\langle\varepsilon_{kl}\rangle_m=$$

$$=\frac{1}{2}C^m_{ijkl}\left[\nabla\langle\mathbf{u}\rangle_m+(\nabla\langle\mathbf{u}\rangle_m)^T+\frac{1}{\Delta\Omega}\int\limits_{\partial S_{wm}}\mathbf{u}\overrightarrow{ds}_m+\left(\frac{1}{\Delta\Omega}\int\limits_{\partial S_{wm}}\mathbf{u}\overrightarrow{ds}_m\right)^T\right],$$

while in the particulate phase $\langle\sigma_{ij}\rangle_p$ is

$$\langle\sigma_{ij}\rangle_p=\langle C^p_{ijkl}\varepsilon^p_{kl}\rangle_p=C^p_{ijkl}\langle\varepsilon_{kl}\rangle_p=$$

$$=\frac{1}{2}C^p_{ijkl}\left[\nabla\langle\mathbf{u}\rangle_p+\left(\nabla\langle\mathbf{u}\rangle_p\right)^T+\frac{1}{\Delta\Omega}\int\limits_{\partial S_{wp}}\mathbf{u}\overrightarrow{ds}_p+\left(\frac{1}{\Delta\Omega}\int\limits_{\partial S_{wp}}\mathbf{u}\overrightarrow{ds}_p\right)^T\right].$$

From this we can try to extract the effective stiffness coefficients C^*_{ijkl} for the Upper scale two-phase averaged stress $\langle\sigma_{ij}\rangle$

$$C^{*}_{ijkl}\langle \varepsilon_{kl}\rangle = C^{*}_{ijkl}\tfrac{1}{2}\nabla\Big(\big(\langle \mathbf{u}\rangle_m + \langle \mathbf{u}\rangle_p\big) + \big(\nabla\langle \mathbf{u}\rangle_m + \nabla\langle \mathbf{u}\rangle_p\big)^T\Big) = \langle \sigma_{ij}\rangle =$$

this left part which is written as in Homogeneous mechanics right side we equalize to the HSP-VAT right part

$$= \tfrac{1}{2}C^{m}_{ijkl}\Big[\nabla\langle \mathbf{u}\rangle_m + (\nabla\langle \mathbf{u}\rangle_m)^T\Big] + \tfrac{1}{2}C^{p}_{ijkl}\Big[\nabla\langle \mathbf{u}\rangle_p + \big(\nabla\langle \mathbf{u}\rangle_p\big)^T\Big] +$$
$$+\tfrac{1}{2}(C^{m}_{ijkl} - C^{p}_{ijkl})\left[\frac{1}{\Delta\Omega}\int_{\partial S_{wm}} \mathbf{u}\overrightarrow{ds}_m + \left(\frac{1}{\Delta\Omega}\int_{\partial S_{wm}} \mathbf{u}\overrightarrow{ds}_m\right)^T\right].$$

When assuming or determined the constant tensor C^{*}_{ijkl} and $\langle \varepsilon_{ij}\rangle = 1$ then we can write (following here [22] homogeneous treatment of heterogeneous problem) assigning the function for $\langle \sigma_{ij}\rangle = C^{*}_{ijkl}$ with $\langle \varepsilon_{ij}\rangle = 1$

$$C^{*}_{ijkl} =$$
$$= \tfrac{1}{2}C^{m}_{ijkl}\Big[\nabla\langle \mathbf{u}\rangle_m + (\nabla\langle \mathbf{u}\rangle_m)^T\Big] + \tfrac{1}{2}C^{p}_{ijkl}\Big[\nabla\langle \mathbf{u}\rangle_p + \big(\nabla\langle \mathbf{u}\rangle_p\big)^T\Big] +$$
$$+\tfrac{1}{2}(C^{m}_{ijkl} - C^{p}_{ijkl})\left[\frac{1}{\Delta\Omega}\int_{\partial S_{wm}} \mathbf{u}\overrightarrow{ds}_m + \left(\frac{1}{\Delta\Omega}\int_{\partial S_{wm}} \mathbf{u}\overrightarrow{ds}_m\right)^T\right].$$

This expression has two additional last terms comparing to usually assumed in the HCM. And even in this simplification it is not equal to what homogeneous elasticity solution methods suggest [22,23] and references therein.

4. EQUILIBRIUM ELASTICITY EQUATION (ELASTOSTATICS)

The equilibrium elasticity equation (Elastostatics) averaged over the two-phase composite material with the accepted ideal junction between phases all over the interface surface will be as following

$$\langle \nabla \cdot \sigma_{ij}\rangle = 0,$$
$$\langle \nabla \cdot \sigma_{ij}\rangle = \nabla \cdot \langle \sigma_{ij}\rangle_m + \frac{1}{\Delta\Omega}\int_{\partial S_{wm}} (\sigma_{ij})\cdot \overrightarrow{ds}_m +$$
$$+\nabla \cdot \langle \sigma_{ij}\rangle_p + \frac{1}{\Delta\Omega}\int_{\partial S_{wp}} (\sigma_{ij})\cdot \overrightarrow{ds}_p =$$
$$= \nabla \cdot \langle \sigma_{ij}\rangle_m + \nabla \cdot \langle \sigma_{ij}\rangle_p + \frac{1}{\Delta\Omega}\int_{\partial S_{wm}} (\sigma_{ij})\cdot \overrightarrow{ds}_m + \frac{1}{\Delta\Omega}\int_{\partial S_{wp}} (\sigma_{ij})\cdot \overrightarrow{ds}_p,$$

we have an elimination of the two integral terms here - those are the traction terms on the interface surfaces ∂S_{wm}, ∂S_{wp} because of the perfect contact conditions on the interface, while those terms won't be eliminated for numerous practical problems,

$$\frac{1}{\Delta\Omega}\int_{\partial S_{wm}}(\sigma_{ij})\cdot\overrightarrow{ds}_m+\frac{1}{\Delta\Omega}\int_{\partial S_{wp}}(\sigma_{ij})\cdot\overrightarrow{ds}_p=0,$$

and finally this simplified Upper scale elasticity equation can be written as

$$\langle\nabla\cdot\sigma_{ij}\rangle=\nabla\cdot\langle\sigma_{ij}\rangle=0,$$

because of equality of interface traction BC (but only if the cracks or other discontinuities, if they exist, are having the equal tractions and normal vectors' $\mathbf{n}_{mi}$ and $\mathbf{n}_{pi}$ are collinear)

$$\frac{1}{\Delta\Omega}\int_{\partial S_{wm}}(\sigma_{ij})\cdot\overrightarrow{ds}_m=\frac{1}{\Delta\Omega}\int_{\partial S_{wm}}(\mathbf{n}_{mi}\cdot\sigma_{ij})ds_m=-\frac{1}{\Delta\Omega}\int_{\partial S_{wp}}(\mathbf{n}_{pi}\cdot\sigma_{ij})ds_p,$$

as soon as

$$\overrightarrow{ds}_m=-\overrightarrow{ds}_p,\quad\frac{1}{\Delta\Omega}\int_{\partial S_{wm}}\mathbf{u}\overrightarrow{ds}_m=-\frac{1}{\Delta\Omega}\int_{\partial S_{wp}}\mathbf{u}\overrightarrow{ds}_p.$$

Further, we can create the Upper scale averaged equilibrium equation in its simplest form as $\nabla\cdot\sigma_{ij}=0,$ which becomes after using the above derived averaged fields of stress via displacement fields in both phases as

$$\nabla\cdot\langle\sigma_{ij}\rangle=0,\Rightarrow$$

or

$$\frac{1}{2}C^m_{ijkl}\nabla\cdot\left[\nabla\langle\mathbf{u}\rangle_m+(\nabla\langle\mathbf{u}\rangle_m)^T\right]+\frac{1}{2}C^p_{ijkl}\nabla\cdot\left[\nabla\langle\mathbf{u}\rangle_p+\left(\nabla\langle\mathbf{u}\rangle_p\right)^T\right]+$$
$$+\frac{1}{2}(C^m_{ijkl}-C^p_{ijkl})\nabla\cdot\left[\frac{1}{\Delta\Omega}\int_{\partial S_{wm}}\mathbf{u}\overrightarrow{ds}_m+\left(\frac{1}{\Delta\Omega}\int_{\partial S_{wm}}\mathbf{u}\overrightarrow{ds}_m\right)^T\right]=0.$$

The last two (four actually) terms are the surficial terms and as such present the tremendous interest for sold state mechanics as long as the physical effect at the interface in homogeneous elastic mechanics do not participate directly in the theory and in modeling. The boundary conditions are the boundary conditions, for the averaged upper scale physical and mathematical modeling even do not have them at all in homogeneous Continuum Mechanics.

The very important point in the derivation of hierarchical upper scale equations in solid state (soft solid state) is that in almost any heterogeneous medium the interface conditions are not perfect, and the material properties formed not only bulk each phase physics, but and in substantial part by the morphology of the polyphasing.

Taking this into account it is imperative to keep all terms in the modeling of upper scale elasticity equations, as for the stationary case in the following equation

$$\langle\nabla\cdot\sigma_{ij}\rangle=\nabla\cdot\langle\sigma_{ij}\rangle_m+\nabla\cdot\langle\sigma_{ij}\rangle_p+$$
$$+\frac{1}{\Delta\Omega}\int_{\partial S_{wm}}(\sigma_{ij})\cdot\overrightarrow{ds}_m+\frac{1}{\Delta\Omega}\int_{\partial S_{wp}}(\sigma_{ij})\cdot\overrightarrow{ds}_p=$$
$$=\frac{1}{2}C^m_{ijkl}\nabla\cdot\left[\nabla\langle\mathbf{u}\rangle_m+(\nabla\langle\mathbf{u}\rangle_m)^T\right]+\frac{1}{2}C^p_{ijkl}\nabla\cdot\left[\nabla\langle\mathbf{u}\rangle_p+\left(\nabla\langle\mathbf{u}\rangle_p\right)^T\right]+$$

$$+\frac{1}{2}(C^m_{ijkl}-C^p_{ijkl})\nabla\cdot\left[\frac{1}{\Delta\Omega}\int\limits_{\partial S_{wm}}\mathbf{u}\overrightarrow{ds}_m+\left(\frac{1}{\Delta\Omega}\int\limits_{\partial S_{wm}}\mathbf{u}\overrightarrow{ds}_m\right)^T\right]+$$
$$+\frac{1}{\Delta\Omega}\int\limits_{\partial S_{wm}}(\sigma_{ij})\cdot\overrightarrow{ds}_m+\frac{1}{\Delta\Omega}\int\limits_{\partial S_{wp}}(\sigma_{ij})\cdot\overrightarrow{ds}_p=0.$$

In this equation which is the "bulk" averaged over the two phases Upper scale heterogeneous media elastostatic equation we can find even larger amount of interface physical effect terms - 4 surficial terms.

And these terms are the main difference and purpose to study, discover and use in heterogeneous polyphase media physical effects that are present, physicists know about most of them, but unattainable neither for physical and mathematical modeling nor for study and simulation via the conventional homogeneous physics tools and methods, as we saw this via numerous examples from a variety of sciences and technologies - http://travkin-hspt.com/compos/index.htm "Composite Engineering"; http://travkin-hspt.com/eldyn/index.htm "Electrodynamics"; http://travkin-hspt.com/fluid/index.htm "Fluid Mechanics"; http://travkin-hspt.com/thermph/index.htm "Thermal Physics"; etc.

The history of mechanics, physical mechanics shows this struggle and search for solution for decades, if not for more than 100 years, since the discovery of the atomic structure of solid matters.

But the most interesting is the interplay of each phase effective coefficients C^{m*}_{ijkl}, C^{p*}_{ijkl} for elasticity problem (meanwhile, those are not the same as the phase coefficients, and even not equal to the homogeneous "effective" coefficients), bulk material effective stiffness coefficient as in this case C^{*}_{ijkl}, and physical meaning and input by the surficial interface physics terms in the governing equations for the upper scale as we have shown in the introduction to this paper, are vital for assessment and design, for example, for the conductivity, diffusivity and momentum problems.

It can be shown how using the same techniques that have been used for the phase upper scale averaged equations one can deduce and use the two-scale mathematical statements for the simulation and assessment of the steady-state two scale elasticity tasks.

5. STATIC ELASTICITY OF SOFT SOLID HETEROGENEOUS POLYPHASE BIOMEDIA AS CYTOSKELETON, CELL, EXTRACELLULAR MATRIX (ECM), TISSUE, ETC.

At present in biomechanics, biophysics application to medicine widely used by most of researchers the homogeneous theories that based on the concept of the homogeneous mixture for biomedia. We won't analyze and observe those approaches here in this paper. There are taken too many of assumptions and conjectures, see analysis in [1-4,17]. We take, instead, the polyphase polyscale HSP-VAT approach in this field.

It is known to continuum mechanics, biophysics researchers that any physical field's within continuum mechanics governing equations deduction has been started with application of the GO theorem to the piece, volume of a material, matter under investigation. We have been researched intensively on this subject in [12]. The hard copy publications [3,4] summarizes shortly the problematic (simply wrong) physics and mathematics of homogeneous elasticity theory used for heterogeneous materials, media.

The matter is that the methods of HSP-VAT give the ability to model, simulate and make calculation practically of an exact nature, as long as we are able to figure out the morphology features, and we can do this at present in biology and medicine research.

When the polyphase HSP-VAT applied to the static elasticity problem for biomedia with fluid component (phase) (m) the equation for averaged over the Upper scale medium stress tensor $\left\langle \sigma_{ij} \right\rangle$ as

$$\langle \sigma_{ij} \rangle = C^m_{ijkl}\left[\nabla \langle \mathbf{V} \rangle_m + (\nabla \langle \mathbf{V} \rangle_m)^T \right] + \frac{1}{2} \sum_{q=1}^{N} C^q_{ijkl} \left[\nabla \langle \mathbf{u} \rangle_q + \left(\nabla \langle \mathbf{u} \rangle_q \right)^T \right] +$$

where we cannot write here the strain tensor ε_{kl} as an averaged $\left\langle \varepsilon_{kj} \right\rangle_q$ already over the phase(s) q or m, because the averaged $\left\langle \varepsilon_{kj} \right\rangle_m$ and $\left\langle \varepsilon_{kj} \right\rangle_q$ means the existence of additional surficial terms

$$+C^m_{ijkl}\left[\frac{1}{\Delta\Omega} \int_{\partial S_{wm}} \mathbf{V} \overrightarrow{ds}_m + \left(\frac{1}{\Delta\Omega} \int_{\partial S_{wm}} \mathbf{V} \overrightarrow{ds}_m \right)^T \right] +$$

$$+\frac{1}{2} \sum_{q=1}^{N} C^q_{ijkl}\left[\frac{1}{\Delta\Omega} \int_{\partial S_{wq}} \mathbf{u} \overrightarrow{ds}_q + \left(\frac{1}{\Delta\Omega} \int_{\partial S_{wq}} \mathbf{u} \overrightarrow{ds}_q \right)^T \right],$$

here used only the deviatory part τ_{ij} of the stress tensor $\sigma^f_{ij} = \left[-p\mathbf{I} + \tau_{ij} \right]$ in the fluid phase where

$$\boldsymbol{\tau}_{ij} = \mu(\nabla \mathbf{V} + (\nabla \mathbf{V})^*) = \mu(2\mathbf{S}_{ij}),$$

where $\mathbf{V}$ is the velocity while the deformation tensor $\mathbf{S}_{ij}$ is

$$\mathbf{S}_{ij} = \frac{1}{2}(\nabla \mathbf{V} + (\nabla \mathbf{V})^*),$$

so in a fluid phase

$$\boldsymbol{\sigma}^f_{ij} = -p\mathbf{I} + \mu(\nabla \mathbf{V} + (\nabla \mathbf{V})^*),$$

and where $C^m_{ijkl} = \mu$ (dynamic viscosity) and C^q_{ijkl} are the stiffness tensors for phases q, when m is the subscript designating the liquid "matrix" or water solution (cytosol) medium volume all over the selected part within the biomedium, that is taken here for simplicity as the Newtonian fluid, while N is the number of elastic parts, organelles, for example, or macromolecular complexes, etc. To take into account the polyphase nature of cytosol there should be taken the one more scale down physics model.

We have in all phase averaged equation here the *2(N+1)* additional surficial terms - that is, the correct averaging for Heterogeneous media brings in the substantially different from the homogeneous elasticity equations.

For further one needs to turn to the static elasticity equation (elastostatics used here as the simplest form elasticity mathematical equation) for the selected part (phase) within the biomedium

$$\nabla \bullet \sigma_{ij} = 0,$$

as in a polyphase medium. The modeling elasticity phenomena for the heterogeneous part of a biomedium demands the elastodynamics governing equations. The Upper scale equilibrium averaged over the two-phase *m* and *N=1*, *p*-phase where the perfect BC for a displacement at the interface has been applied (just as an example) to the biomedium, will be in its simplest form $\langle \nabla \sigma_{ij} \rangle = 0$, which becomes after using the above derived averaged fields of stress via the displacement field in soft solids and velocity field in a fluid phase as

$$\mu \nabla \bullet \left[\nabla \langle \mathbf{V} \rangle_m + \left(\nabla \langle \mathbf{V} \rangle_m \right)^T \right] + \tfrac{1}{2} C_{ijkl}^p \nabla \bullet \left[\nabla \langle \mathbf{u} \rangle_p + \left(\nabla \langle \mathbf{u} \rangle_p \right)^T \right] +$$

$$+ \mu \nabla \bullet \left[\frac{1}{\Delta\Omega} \int_{\partial S_{wm}} \mathbf{V} \overrightarrow{ds}_m + \left(\frac{1}{\Delta\Omega} \int_{\partial S_{wm}} \mathbf{V} \overrightarrow{ds}_m \right)^T \right] +$$

$$+ \tfrac{1}{2} C_{ijkl}^p \nabla \bullet \left[\frac{1}{\Delta\Omega} \int_{\partial S_{wp}} \mathbf{u} \overrightarrow{ds}_p + \left(\frac{1}{\Delta\Omega} \int_{\partial S_{wp}} \mathbf{u} \overrightarrow{ds}_p \right)^T \right] -$$

$$- \nabla \langle p \rangle_m - \frac{1}{\Delta\Omega} \int_{\partial S_{wm}} p \overrightarrow{ds}_m = 0.$$

The phenomena of viscoelasticity that are truly pertained to the some part-subvolumes of interior of a biomedium like cell's skeleton, to the cell itself, to the cytoplasm as a whole for certain microorganisms and for locomotion, to the ECM, etc. must be accounted for by the two-scale, at least, models. The well studied and failed tasks in these fields are the viscoelastic properties of blood polyphase fluid, blood flow, other biomedia models, that are also the subject for scaled, polyphase, polyphysics concepts, modeling which should be followed by the scaled simulation. The basic concepts and viscoelastic, viscoplastic governing equations for heterogeneous media via HSP-VAT also have been developed in the 90s.

6. ELASTODYNAMICS --- THE WAVE EQUATIONS IN HETEROGENEOUS SOLID STATE MEDIA

We would start with the homogeneous media transient linear elasticity equation which reads as

$$\nabla \bullet \sigma_{ij} + \mathbf{F} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2},$$

where **F** - is the body force per unit volume [(N/(m^3))].

Averaged over the REV for the upper scale presentation using the developed HSP-VAT methods for appropriate operators having place in the homogeneous equation brings this equation to the form

$$\langle \nabla \bullet \sigma_{ij} \rangle + \langle \mathbf{F} \rangle = \left\langle \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} \right\rangle,$$

$$\langle \nabla \bullet \sigma_{ij} \rangle + \langle \mathbf{F} \rangle = \nabla \bullet \langle \sigma_{ij} \rangle_m + \frac{1}{\Delta\Omega} \int\limits_{\partial S_{wm}} (\sigma_{ij}) \bullet \overrightarrow{ds}_m +$$

$$+\nabla \bullet \langle \sigma_{ij} \rangle_p + \frac{1}{\Delta\Omega} \int\limits_{\partial S_{wp}} (\sigma_{ij}) \bullet \overrightarrow{ds}_p + \langle \mathbf{F} \rangle =$$

$$= \nabla \bullet \langle \sigma_{ij} \rangle_m + \nabla \bullet \langle \sigma_{ij} \rangle_p + \frac{1}{\Delta\Omega} \int\limits_{\partial S_{wm}} (\sigma_{ij}) \bullet \overrightarrow{ds}_m +$$

$$+\frac{1}{\Delta\Omega} \int\limits_{\partial S_{wp}} (\sigma_{ij}) \bullet \overrightarrow{ds}_p + \langle \mathbf{F} \rangle = \left\langle \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} \right\rangle =$$

$$= \rho_m \frac{\partial^2 \langle \mathbf{u}_m \rangle_m}{\partial t^2} - \rho_m \frac{\partial}{\partial t} \left[\frac{1}{\Delta\Omega} \int\limits_{\partial S_{wm}} (\mathbf{V}_{sq}(\mathbf{u}_m)) \bullet \vec{ds}_m \right] -$$

$$-\frac{\rho_m}{\Delta\Omega} \int\limits_{\partial S_{wm}} \left(\mathbf{V}_{sq} \left(\frac{\partial}{\partial t} (\mathbf{u}_m) \right) \right) \bullet \vec{ds}_m +$$

$$+\rho_p \frac{\partial^2 \langle \mathbf{u}_p \rangle_p}{\partial t^2} - \rho_p \frac{\partial}{\partial t} \left[\frac{1}{\Delta\Omega} \int\limits_{\partial S_{wp}} (\mathbf{V}_{sq}(\mathbf{u}_p)) \bullet \vec{ds}_p \right] -$$

$$-\frac{\rho_p}{\Delta\Omega} \int\limits_{\partial S_{wp}} \left(\mathbf{V}_{sq} \left(\frac{\partial}{\partial t} (\mathbf{u}_p) \right) \right) \bullet \vec{ds}_p, \qquad \left[\frac{N}{m^3} \right],$$

where $\mathbf{V}_{sq}$ is the speed of displacement of q-th phase interface surface, $\sum_{q=1}^{N} (\partial S_{wq}) = \partial S_{wm}$ in this equation for matrix and particulate phase elasticity. Well, combining all these $\left(\langle\ \rangle_m + \sum_{q=1}^{N} \langle\ \rangle_q \right)$ separate terms averaged mathematics we got for the two-phase *m* and *p* (*q*=1) the averaged equation for upper scale

$$\frac{1}{2} C^m_{ijkl} \nabla \bullet \left[\nabla \langle \mathbf{u} \rangle_m + (\nabla \langle \mathbf{u} \rangle_m)^T \right] + \frac{1}{2} C^p_{ijkl} \nabla \bullet \left[\nabla \langle \mathbf{u} \rangle_p + \left(\nabla \langle \mathbf{u} \rangle_p \right)^T \right] +$$

$$+\frac{1}{2} (C^m_{ijkl} - C^p_{ijkl}) \nabla \bullet \left[\frac{1}{\Delta\Omega} \int\limits_{\partial S_{wm}} \mathbf{u} \overrightarrow{ds}_m + \left(\frac{1}{\Delta\Omega} \int\limits_{\partial S_{wm}} \mathbf{u} \overrightarrow{ds}_m \right)^T \right] +$$

$$+\frac{1}{\Delta\Omega}\int_{\partial S_{wm}}(\sigma_{ij})\cdot\overrightarrow{ds}_m+\frac{1}{\Delta\Omega}\int_{\partial S_{wp}}(\sigma_{ij})\cdot\overrightarrow{ds}_p+\langle\mathbf{F}\rangle_m+\langle\mathbf{F}\rangle_p=$$

$$=(\rho_m+\rho_p)\frac{\partial^2\langle\mathbf{u}_m\rangle_m}{\partial t^2}-(\rho_m-\rho_p)\frac{\partial}{\partial t}\left[\frac{1}{\Delta\Omega}\int_{\partial S_{wm}}(\mathbf{V}_{sq}(\mathbf{u}_m))\cdot\vec{ds}_m\right]-$$

$$-(\rho_m-\rho_p)\frac{1}{\Delta\Omega}\int_{\partial S_{wm}}\left(\mathbf{V}_{sq}\left(\frac{\partial}{\partial t}(\mathbf{u}_m)\right)\right)\cdot\vec{ds}_m,$$

this equation is approximately valid only if the BC for displacement fields on the interface ∂S_{wm} are as following

$$\vec{ds}_m=-\vec{ds}_p,\quad \mathbf{u}_m=\mathbf{u}_p,\quad \text{but}\;\; \mathbf{n}_m\cdot\boldsymbol{\sigma}|_{\partial S_{wm}}\neq\mathbf{n}_p\cdot\boldsymbol{\sigma}|_{\partial S_{wp}}.$$

In this equation which is the Upper scale averaged over the two phases elastodynamics equation we can find even bigger amount of interface physical effect terms - 6 surficial terms.

Especially troublesome the upper scale EE will be when the imperfect BC of displacements $\mathbf{u}|_{\partial S_{wm}}\neq\mathbf{u}|_{\partial S_{wp}}$ and traction vectors $\mathbf{n}_m\cdot\sigma|_{\partial S_{wm}}\neq\mathbf{n}_p\cdot\sigma|_{\partial S_{wp}}$ satisfied (that is the practical problems of usual outcome), in this equation which is the "bulk" averaged over the two phases elastodynamics equation we can find even bigger amount of interface physical effect terms - 10 surficial terms with the different displacement vector field **u** and stress tensor fields (σ_{ij}) should be enforced as in the following

$$\frac{1}{2}C^m_{ijkl}\nabla\cdot\left[\nabla\langle\mathbf{u}\rangle_m+(\nabla\langle\mathbf{u}\rangle_m)^T\right]+\frac{1}{2}C^p_{ijkl}\nabla\cdot\left[\nabla\langle\mathbf{u}\rangle_p+\left(\nabla\langle\mathbf{u}\rangle_p\right)^T\right]+$$

$$+\frac{1}{2}(C^m_{ijkl})\nabla\cdot\left[\frac{1}{\Delta\Omega}\int_{\partial S_{wm}}\mathbf{u}\overrightarrow{ds}_m+\left(\frac{1}{\Delta\Omega}\int_{\partial S_{wm}}\mathbf{u}\overrightarrow{ds}_m\right)^T\right]+$$

$$+\frac{1}{2}(C^p_{ijkl})\nabla\cdot\left[\frac{1}{\Delta\Omega}\int_{\partial S_{wp}}\mathbf{u}\overrightarrow{ds}_p+\left(\frac{1}{\Delta\Omega}\int_{\partial S_{wp}}\mathbf{u}\overrightarrow{ds}_p\right)^T\right]+$$

$$+\frac{1}{\Delta\Omega}\int\limits_{\partial S_{wm}}(\sigma_{ij})\cdot\overrightarrow{ds}_m+\frac{1}{\Delta\Omega}\int\limits_{\partial S_{wp}}(\sigma_{ij})\cdot\overrightarrow{ds}_p+\langle\mathbf{F}\rangle_m+\langle\mathbf{F}\rangle_p=$$

$$=(\rho_m)\frac{\partial^2\langle\mathbf{u}\rangle_m}{\partial t^2}+(\rho_p)\frac{\partial^2\langle\mathbf{u}\rangle_p}{\partial t^2}-(\rho_m)\frac{\partial}{\partial t}\left[\frac{1}{\Delta\Omega}\int\limits_{\partial S_{wm}}(\mathbf{V}_{sq}(\mathbf{u}_m))\cdot\vec{ds}_m\right]-$$

$$-(\rho_p)\frac{\partial}{\partial t}\left[\frac{1}{\Delta\Omega}\int\limits_{\partial S_{wp}}(\mathbf{V}_{sq}(\mathbf{u}_p))\cdot\vec{ds}_p\right]-$$

$$-\frac{(\rho_m)}{\Delta\Omega}\int\limits_{\partial S_{wm}}\left(\mathbf{V}_{sq}\left(\frac{\partial}{\partial t}(\mathbf{u}_m)\right)\right)\cdot\vec{ds}_m-$$

$$-\frac{(\rho_p)}{\Delta\Omega}\int\limits_{\partial S_{wp}}\left(\mathbf{V}_{sq}\left(\frac{\partial}{\partial t}(\mathbf{u}_p)\right)\right)\cdot\vec{ds}_p.$$

Completely different amount of possibilities and the new physical effects right now is presented (after 17-18 years of waiting period whether other workers might be able to advance the Ht Continuum Mechanics toward heterogeneous treatment techniques) for researchers in HtCM.

7. MASS AND MOMENTUM TRANSPORT IN POROUS PART OF SOFT BIOMEDIA

In homogenous media the physical and mathematical models used by theoretical biologists and workers from all over medical sciences are taken from conventional Homogeneous physical disciplines. There are many thousands of this kind of studies exist. We would be concerned here only with a few well known in heterogeneous physics methods for description of these enumerated above transport phenomena for biomedia, which is simply more complicated media and phenomena in those, but the basis of modeling should be the same for any heterogeneous processes because mathematics is the universal language science.

We have done substantial research and writing in the 80-90s in these Continuum Mechanics disciplines for HSP-VAT clarification. The theoretical developments were suggested and many problems were solved firstly as the two-scale ones, even analytically, for example in [11]. Summaries and further ideas with progress in theories and solutions while communicating to other physical sciences can be found in our [45] with cited publications.

The transport momentum equations within the separate porous part of biomedium's region is based on a creep flow using the fluid dynamics laminar flow equations with the permeable interface surface ∂S_w between the phases (for the two phases), or a number of interface surfaces ∂S_{wq} for multiple phases accounted for.

The half-linear momentum linear equations for the each "fluid" phase at the Lower still continuum scale have the form

$$\nabla\mathbf{V}=0,$$

$$\varrho_f\left(\frac{\partial \mathbf{V}}{\partial t}+\mathbf{V}\bullet\nabla\mathbf{V}\right)=-\nabla p+\mu\nabla^2\mathbf{V},$$

where the nonlinear term ($\mathbf{V}\bullet\nabla\mathbf{V}$) can be easily dropped in the momentum equation for some cases except strictly for some blood flow in a larger vasculature vessels.
Now, one can write down the Upper scale averaged "fluid" dynamics equations. The first is

$$\nabla\langle\mathbf{V}\rangle_f+\frac{1}{\Delta\Omega}\int_{\partial S_w} U_i\ \bullet d\vec{s}=0,$$

with the second filtration term; while the momentum equation for the "fluid" phase (cytosol's part, for example)

$$\varrho_f\left(\langle m\rangle\frac{\partial\widetilde{\mathbf{V}}}{\partial t}-\frac{1}{\Delta\Omega}\int_{\partial S_w}(\mathbf{V}_s\mathbf{V})\bullet d\vec{s}+\langle m\rangle\widetilde{\mathbf{V}}\bullet\nabla\widetilde{\mathbf{V}}-\widetilde{\mathbf{V}}\frac{1}{\Delta\Omega}\int_{\partial S_w}\mathbf{V}\bullet d\vec{s}+\right.$$

$$\left.+\nabla<\widehat{U}_i\widehat{U}_i>_f+\frac{1}{\Delta\Omega}\int_{\partial S_w}\mathbf{V}\mathbf{V}\bullet d\vec{s}\right)=$$

$$=-\nabla(\langle m\rangle\widetilde{p})-\frac{1}{\Delta\Omega}\int_{\partial S_w}p\ d\vec{s}+\mu\nabla\bullet\nabla\left(\langle m\rangle\widetilde{\mathbf{V}}\right)+$$

$$+\mu\nabla\bullet\left[\frac{1}{\Delta\Omega}\int_{\partial S_w}\mathbf{V}\bullet d\vec{s}\right]+\frac{\mu}{\Delta\Omega}\int_{\partial S_w}\nabla\mathbf{V}\bullet d\vec{s}.$$

Of course, to these two-scale equations one must add others for the energy and/or heat (other character) transport for each of the "phase", whatever is more relevant to the studied task.
These types and great number of modeling heterogeneous mathematical equations are not known for use in biology. Unfortunately, the education and other obvious reasons hinder the penetration of these modeling concepts to knowledge base of biological and medicine practices.
One of the latest examples is the series of models suggested and studied in the monograph [46] where in the first most realistic and advanced chapter by Nakayama et al. [47] one can find the sets of governing equations that cannot be found as the correct modeling equations and as well the modeling equations for heterogeneous (porous) media in the other whole book chapters.

8. POROELASTICITY OF SOFT BIOMEDIA, POLYMERS WITH FLUID MECHANICS IN THE BIOPOROUS TWO-SCALE MEDIA

To present the two-phase two-scale bioporous media governing equations for both scales, because the one scale modeling and data reduction won't be able to find the Upper scale - whta usually researchers need indeed, while the modeling and simulation for both scale is the only solution that can reveal the interconnection of local and non-local, sometimes depending on boundary conditions on the Upper and sometimes the primary interest lays with the input from Lower scale boundary conditions.

The initial governing equations for incompressible fluid at the lower scale homogeneous media

$$\nabla \mathbf{V} = 0, \quad in\ \Omega_f,$$

$$\varrho_f \left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} \right) = -\nabla p + \mu \nabla^2 \mathbf{V},$$

followed from

$$\sigma_{ij}^{f} = -p\mathbf{I} + \mu \nabla \mathbf{V},\ in\ \Omega_f,$$

where also the boundary conditions at the lower scale interface

$$\mathbf{V} = \frac{\partial \mathbf{u}}{\partial t},\ on\ \partial\Omega_f = \partial\Omega_s = \partial S_w,$$

$$\mathbf{n} \cdot \sigma_{ij}^{f} = \mathbf{n} \cdot \sigma_{ij}^{s},\ on\ \partial\Omega_f = \partial\Omega_s,$$

while in the soft solid homogeneous phase the elasticity homogeneous equation is

$$\langle \nabla \cdot \sigma_{ij}^{s} \rangle + \langle \mathbf{F} \rangle = \left\langle \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} \right\rangle, \quad in\ \Omega_s.$$

Using the above advanced upper scale governing equations one has for fluid and solid phase equations of flow and conjugated elasticity equation

$$\nabla \langle \mathbf{V} \rangle_f + \frac{1}{\Delta\Omega} \int_{\partial S_w} U_i \cdot d\vec{s} = 0,$$

$$\varrho_f \left(\langle m \rangle \frac{\partial \widetilde{\mathbf{V}}}{\partial t} - \frac{1}{\Delta\Omega} \int_{\partial S_w} (\mathbf{V}_s \mathbf{V}) \cdot d\vec{s} + \langle m \rangle \widetilde{\mathbf{V}} \cdot \nabla \widetilde{\mathbf{V}} - \widetilde{\mathbf{V}} \frac{1}{\Delta\Omega} \int_{\partial S_w} \mathbf{V} \cdot d\vec{s} + \right.$$

$$\left. + \nabla < \widehat{U}_i \widehat{U}_i >_f + \frac{1}{\Delta\Omega} \int_{\partial S_w} \mathbf{V}\mathbf{V} \cdot d\vec{s} \right) =$$

$$= -\nabla(\langle m \rangle \widetilde{p}) - \frac{1}{\Delta\Omega} \int_{\partial S_w} p\ d\vec{s} + \mu \nabla \cdot \nabla \left(\langle m \rangle \widetilde{\mathbf{V}} \right) +$$

$$+ \mu \nabla \cdot \left[\frac{1}{\Delta\Omega} \int_{\partial S_w} \mathbf{V} \cdot d\vec{s} \right] + \frac{\mu}{\Delta\Omega} \int_{\partial S_w} \nabla \mathbf{V} \cdot d\vec{s}.$$

and in soft solid phase (matrix)

$$\frac{1}{2} C_{ijkl}^{m} \nabla \cdot \left[\nabla \langle \mathbf{u} \rangle_m + \left(\nabla \langle \mathbf{u} \rangle_m \right)^T \right] +$$

$$+\frac{1}{2}\left(C_{ijkl}^{m}\right)\nabla\bullet\left[\frac{1}{\Delta\Omega}\int_{\partial S_{wm}}\mathbf{u}\overrightarrow{ds}_{m}+\left(\frac{1}{\Delta\Omega}\int_{\partial S_{wm}}\mathbf{u}\overrightarrow{ds}_{m}\right)^{T}\right]+$$

$$+\frac{1}{\Delta\Omega}\int_{\partial S_{wm}}\left(\sigma_{ij}\right)\bullet\overrightarrow{ds}_{m}+\langle\mathbf{F}\rangle_{m}=$$

$$=\left(\rho_{m}\right)\frac{\partial^{2}\langle\mathbf{u}\rangle_{m}}{\partial t^{2}}-\left(\rho_{m}\right)\frac{\partial}{\partial t}\left[\frac{1}{\Delta\Omega}\int_{\partial S_{wm}}\left(\mathbf{V}_{sq}\left(\mathbf{u}_{m}\right)\right)\bullet\vec{ds}_{m}\right]-$$

$$-\frac{\left(\rho_{m}\right)}{\Delta\Omega}\int_{\partial S_{wm}}\left(\mathbf{V}_{sq}\left(\frac{\partial}{\partial t}\left(\mathbf{u}_{m}\right)\right)\right)\bullet\vec{ds}_{m}.$$

This set of general two-scale two-phase equations for poroelastic momentum transport is published for the first time, meanwhile this modeling statement is the one so important that without it correctly cannot be studied most of the biomedia dynamics problems - neither acoustics nor any dynamics of air, blood or lymph in human tissues, organs. In other our works, publications these 6 two-scale two-phase fluid mechanics, poroelasticity governing equations can be found in numerous versions and application solutions. These equations should be considered as the initial principal set of equations for the two-scale two-phase fluid mechanics, poroelasticity phenomena in porous solids, porous soft solids, in any biomedium that is not completely fluid. The first versions of this theory have been developed in the 90s.

9. CONCLUSIONS

We have shown in this paper for the first time the obtained in 1994-95 short, but rather full path to derivation of the two-scale (local-nonlocal) mathematical formulation for the porous phase fluid transport and elasticity in the soft solid phase of heterogeneous, polyphase biomedia.
Obtained the two-scale models are much more complex than the homogeneous one scale media poroelasticity mathematical models and their statements. Nonetheless, they are only correct for Heterogeneous media, more physically descriptive and useful as it is shown during the many years developments in other physical disciplines such as fluid mechanics, thermal physics, acoustics, electrodynamics, heterogeneous mathematics, atomic scale physics, nuclear physics, and their numerous applications in engineering sciences, biological sciences, technologies.
Here is presented as the two-scale, at least, models interconnected narrative, where the Lower scale models are of familiar from the major physical disciplines description with the homogeneous fields that are formulated as homogeneous mathematical statements. Meanwhile, the upper second scale modeling formulation for the porous media fluid flow and elasticity physical processes mentioned above have been developed for many years already, used for studies, obtained the results unattainable with the one scale homogeneous physics.

Now the tools and methods of HSP-VAT are available in the area of conjugated fluid flow and poroelasticity in heterogeneous and scaled media as solid state and soft solid matter, biomedia.

10. ACKNOWLEDGEMENTS

We would like to sincerely express our acknowledgment to T.Champion for help in providing this work.

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